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# Solution of the linear and non-linear differential equations by using Homotopy perturbation method

KEYWORDS: System of linear ordinary differential equations; Abelian differential equations; Homotopy perturbation method; Numerical simulation.

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**ABSTRACT** *In this paper we use He's Homotopy perturbation method is applied to solve a system of linear ordinary differential equations of the first order and some first order non-linear ordinary differential equations like Abelian differential equations. The method yields solutions in convergent series form with easily computable terms. The result shows that these methods are very convenient and can be applied to a large class of problems. Some numerical examples are given to the effectiveness of the method. Our analytical results are compared with the numerical results and a satisfactory agreement is noted.*

## 1.INTRODUCTION

A system of ordinary differential equations of the first order can be considered as [1-4]:

$$\begin{aligned} y_1' &= f_1(x, y_1, \ldots y_n) \\ y_2' &= f_2(x, y_1, \ldots y_n) \\ &. \\ &. \\ &. \\ y_n' &= f_n(x, y_1, \ldots, y_n) \end{aligned} \quad (1)$$

where each equation represents the first derivative of one of the unknown functions as a mapping depending on the independent variable $x$, and $n$ unknown functions $f_1, f_1, \ldots, f_n$. Since every ordinary differential equation of order $n$ can be written as a system consisting of $n$ ordinary differential equation of order one, we restrict our study to a system of differential equations of the first order.

Linear and non-linear phenomena are of fundamental importance in various fields of science and engineering. Most models of real – life problems are still very difficult to solve. Therefore, approximate analytical solutions such as Homotopyperturbation method (HPM) [5-16] were introduced. This method is the most effective and convenient ones for both linear and non-linear equations. Perturbation method is based on assuming a small parameter. The majority of non-linear problems, especially those having strong non-linearity, have no small parameters at all and the approximate solutions obtained by the perturbation methods, in most cases, are valid only for small values of the small parameter. Generally, the perturbation solutions are uniformly valid as long as a scientific system parameter is small. However, we cannot rely fully on the approximations, because there is no criterion on which the small parameter should exists. Thus, it is essential to check the validity of the approximations numerically and/or experimentally. To overcome these difficulties, HPM have been proposed recently.

Recently, many authors have applied the Homotopy perturbation method (HPM) to solve the non-linear boundary value problem in physics and engineering sciences [5-8]. Recently this method is also used to solve some of the non-linear problem in physical sciences [9-11]. This method is a combination of Homotopy in topology and classic perturbation techniques. Ji-Huan He used to solve the Light hill equation [8], the Diffusion equation [9] and the Blasius equation [10-11]. The HPM is unique in its applicability, accuracy and efficiency. The HPM uses the imbedding parameter $p$ as a small parameter, and only a few iterations are needed to search for an asymptotic solution.

## 2. Basic concepts of the Homotopy perturbation method [5-16]

To explain this method, let us consider the following function:

$$D_o(u) - f(r) = 0, \quad r \in \Omega \quad (A.1)$$

with the boundary conditions of

$$B_o(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma \quad (A.2)$$

where $D_o$ is a general differential operator, $B_o$ is a boundary operator, $f(r)$ is a known analytical function and $\Gamma$ is the boundary of the domain $\Omega$. In general, the operator $D_o$ can be divided into a linear part $L$ and a non-linear part $N$. Equation (A. 1) can therefore be written as

$$L(u) + N(u) - f(r) = 0 \quad (A.3)$$

By the Homotopy technique, we construct a Homotopy $v(r, p): \Omega \times [0,1] \to \Re$ that satisfies

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[D_o(v) - f(r)] = 0 \quad (A.4)$$

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \quad (A.5)$$

where $p \in [0, 1]$ is an embedding parameter, and $u_0$ is an initial approximation of eqn. (A.1) that satisfies the boundary conditions. From the eqns. (A.4) and (A.5), we have

$$H(v,0) = L(v) - L(u_0) = 0 \quad (A.6)$$

$$H(v,1) = D_o(v) - f(r) = 0 \quad (A.7)$$

When $p$=0, the eqns. (A. 4) and (A. 5) become linear equations. When $p$ =1, they become non-linear equations. The process of changing $p$ from zero to unity is that of $L(v) - L(u_0) = 0$ to $D_o(v) - f(r) = 0$. We first use the embedding parameter $p$ as a "small parameter" and assume that the solutions of

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the eqns. (A. 4) and (A. 5) can be written as a power series in $p$ :

$$v = v_0 + pv_1 + p^2 v_2 + ... \quad (A.8)$$

Setting $p = 1$ results in the approximate solution of the eqn. (A.1):

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + ... \quad (A.9)$$

This is the basic idea of the HPM.

## 3. Numerical examples

In this part we present three examples. The first example is considered to illustrate the method for solving a system of linear ordinary differential equations of orders one. While in the second and the third examples, we solve first order non-linear differential equations namely the Abelian differential equations.

**Example: 1**

$$\frac{dy_1}{dx} - y_3 + \cos x = 0 \quad (1.1)$$

$$\frac{dy_2}{dx} - y_3 + e^x = 0 \quad (1.2)$$

$$\frac{dy_3}{dx} - y_1 + y_2 = 0 \quad (1.3)$$

with the initial conditions

$$y_1(0) = 1 \; y_2(0) = 0 \text{ and } y_3(0) = 2 \quad (1.4)$$

We construct the Homotopy as follows:

$$(1-p)\left[\frac{dy_1}{dx} + \cos x\right] + p\left[\frac{dy_1}{dx} - y_3 + \cos x\right] = 0 \quad (1.5)$$

$$(1-p)\left[\frac{dy_2}{dx} + e^x\right] + p\left[\frac{dy_2}{dx} - y_3 + e^x\right] = 0 \quad (1.6)$$

$$(1-p)\left[\frac{dy_3}{dx}\right] + p\left[\frac{dy_3}{dx} - y_1 + y_2\right] = 0 \quad (1.7)$$

The analytical solution of the eqn. (1.4) is

$$y_1 = y_{10} + py_{11} + p^2 y_{12} + ... \quad (1.8)$$

Similarly the analytical solution of the eqns. (1.5)-(1.6) as follows

$$y_2 = y_{20} + py_{21} + p^2 y_{22} + ... \quad (1.9)$$

$$y_3 = y_{30} + py_{31} + p^2 y_{32} + ... \quad (1.10)$$

Substituting the eqn. (1.8) into an eqn. (1.5), we get

$$(1-p)\left[\frac{d(y_{10} + py_{11} + p^2 y_{12} + ...)}{dx} + \cos x\right] + p\left[\begin{array}{l}\frac{d(y_{10} + py_{11} + p^2 y_{12} + ...)}{dx} \\ -(y_{30} + py_{31} + p^2 y_{32} + ...) + \cos x\end{array}\right] = 0 \quad (1.11)$$

Substituting the eqn. (1.9) into an eqn. (1.6), we get

$$(1-p)\left[\frac{d(y_{20} + py_{21} + p^2 y_{22} + ...)}{dx} + e^x\right] + p\left[\begin{array}{l}\frac{d(y_{20} + py_{21} + p^2 y_{22} + ...)}{dx} \\ -(y_{30} + py_{31} + p^2 y_{32} + ...) + e^x\end{array}\right] = 0 \quad (1.12)$$

Substituting the eqn. (1.10) into the eqn. (1.7), we get

$$(1-p)\left[\frac{d(y_{30} + py_{31} + p^2 y_{32} + ...)}{dx}\right] + p\left[\begin{array}{l}\frac{d(y_{30} + py_{31} + p^2 y_{32} + ...)}{dx} \\ -(y_{10} + py_{11} + p^2 y_{12} + ...) \\ +(y_{20} + py_{21} + p^2 y_{22} + ...)\end{array}\right] = 0 \quad (1.13)$$

Comparing the coefficients of like power $p$ in the eqns. (1.11)-(1.13), we get

$$p^0 : \frac{dy_{10}}{dx} + \cos x = 0 \quad (1.14)$$

$$p^0 : \frac{dy_{20}}{dx} + e^x = 0 \quad (1.15)$$

$$p^0 : \frac{dy_{30}}{dx} = 0 \quad (1.16)$$

$$p^1 : \frac{dy_{11}}{dx} - y_{30} = 0 \quad (1.17)$$

$$p^1 : \frac{dy_{21}}{dx} - y_{30} = 0 \quad (1.18)$$

$$p^1 : \frac{dy_{31}}{dx} - y_{10} + y_{20} = 0 \quad (1.19)$$

$$p^2 : \frac{dy_{12}}{dx} - y_{31} = 0 \quad (1.20)$$

$$p^2 : \frac{dy_{22}}{dx} - y_{31} = 0 \quad (1.21)$$

$$p^2 : \frac{dy_{32}}{dx} - y_{11} + y_{21} = 0 \quad (1.22)$$

The initial approximations are as follows

$$y_{10}(0) = 1, \quad y_{20}(0) = 0 \text{ and } y_{30}(0) = 2 \quad (1.23)$$

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$$y_{1i}(0)=0 \;\; y_{2i}(0)=0 \text{ and } y_{3i}(0)=0, i=1,2,3,.. \tag{1.24}$$

Solving the eqns. (1.14)-(1.22) and using the initial conditions eqns. (1.23)-(1.24) we obtain the following results:

$$y_{10}=-\sin x+1 \tag{1.25}$$
$$y_{11}=2x \tag{1.26}$$
$$y_{12}=\sin x+e^{x}-2x-1 \tag{1.27}$$
$$y_{20}=-e^{x}+1 \tag{1.28}$$
$$y_{21}=2x \tag{1.29}$$
$$y_{22}=\sin x+e^{x}-2x-1 \tag{1.30}$$
$$y_{30}=2 \tag{1.31}$$
$$y_{31}=\cos x+e^{x}-2 \tag{1.32}$$
$$y_{32}=0 \tag{1.33}$$

According to the HPM, we can conclude that

$$y_1=\lim_{p\to1} y_1(x)=y_{10}+\mathrm{p}y_{11}+\mathrm{p}^2y_{12} \tag{1.34}$$
$$y_2=\lim_{p\to1} y_2(x)=y_{20}+\mathrm{p}y_{21}+\mathrm{p}^2y_{22} \tag{1.35}$$
$$y_3=\lim_{p\to1} y_3(x)=y_{30}+\mathrm{p}y_{31}+\mathrm{p}^2y_{32} \tag{1.36}$$

After putting the eqns. (1.25)-(1.27) into the eqn. (1.34),the eqns. (1.28)-(1.30)into the eqn. (1.35) and the eqns. (1.31)- (1.33) into the eqn. (1.36) respectively, we obtain the solutions.

$$y_1(x)=e^{x} \tag{1.37}$$
$$y_2(x)=\sin x \tag{1.38}$$
$$y_3(x)=\cos x+e^{x} \tag{1.39}$$

**Example: 2**

Consider the following system of differential equations [3-4]

$$\frac{dy}{dx}-1+y-y^3=0 \tag{2.1}$$

with the initial condition

$$y(0)=0 \tag{2.2}$$

We construct the Homotopy as follows:

$$(1-p)\left[\frac{dy}{dx}-1\right]+p\left[\frac{dy}{dx}-1+y-y^3\right]=0 \tag{2.3}$$

The analytical solution of the eqn. (2.3) is

$$y=y_0+py_1+p^2y_2+... \tag{2.4}$$

Substituting the eqn.(2.4) into the eqn. (2.3), we get

$$(1-p)\left[\frac{d\left(y_0+py_1+p^2y_2+...\right)}{dx}-1\right] + \left[\begin{array}{l}\frac{d\left(y_0+py_1+p^2y_2+...\right)}{dx}-1\\ +\left(y_0+py_1+p^2y_2+...\right)\\ -\left(y_0+py_1+p^2y_2+...\right)^3\end{array}\right]=0 \tag{2.5}$$

Comparing the coefficients of like power p in the eqn. (2.5), we get

$$p^0:\frac{dy_0}{dx}-1=0 \tag{2.6}$$
$$p^1:\frac{dy_1}{dx}+y-y^3=0 \tag{2.7}$$

The initial approximations are as follows

$$y(0)=0 \tag{2.8}$$
$$y_1(0)=0, i=1,2,3,... \tag{2.9}$$

Solving the eqns. (2.6) and (2.7) and using the initial conditions in the eqns. (2.8)-(2.9) we obtain the following results:

$$y_0(x)=x \tag{2.10}$$
$$y_1(x)=\frac{-x^2}{2}+\frac{x^4}{4} \tag{2.11}$$

According to the HPM, we can conclude that

$$y=\lim_{p\to1} y(x)=y_0+\mathrm{p}y_1+\mathrm{p}^2y_2 \tag{2.12}$$

After putting the eqns. (2.10)-(2.11) into the eqn. (2.12) respectively, we obtain the solutions.

$$y(x)=x-\frac{x^2}{2}+\frac{x^4}{4} \tag{2.13}$$

**Example: 3**

Consider the following system of differential equations [3-4]

$$\frac{dy}{dx}-4-4xy-2xy^2-x^2y^3=0 \tag{3.1}$$

with the initial condition

$$y(0)=0 \tag{3.2}$$

We construct the Homotopy as follows:

$$(1-p)\left[\frac{dy}{dx}-4\right] +p\left[\frac{dy}{dx}-4-4xy-2xy^2-x^2y^3\right]=0 \tag{3.3}$$

The analytical solution of the eqn. (3.3) is

$$y=y_0+py_1+p^2y_2+... \tag{3.4}$$

Substituting the eqn.(3.4) into an eqn. (3.3), we get

$$(1-p)\left[\frac{d\left(y_0+py_1+p^2y_2+...\right)}{dx}-4\right] + p\left[\begin{array}{l}\frac{d\left(y_0+py_1+p^2y_2+...\right)}{dx}-4 \\ -4x\left(y_0+py_1+p^2y_2+...\right) \\ -2x\left(y_0+py_1+p^2y_2+...\right) \\ -x^2\left(y_0+py_1+p^2y_2+...\right)^3\end{array}\right]=0 \quad (3.5)$$

Comparing the coefficients of like power $p$ in the eqn. (3.5), we get

$$p^0:\frac{dy_0}{dx}-4=0 \quad (3.6)$$

$$p^1:\frac{dy_1}{dx}-16x^2-32x^3-64x^6=0 \quad (3.7)$$

The initial approximations are as follows:

$$y(0)=0 \quad (3.8)$$

$$y_1(0)=0, i=1,2,3,... \quad (3.9)$$

Solving the eqns. (3.6) and (3.7) and using the initial conditions the eqns. (3.8)-(3.9) we obtain the following results:

$$y_0(x)=4x \quad (3.10)$$

$$y_1(x)=4x+\frac{16}{3}x^3+8x^4+\frac{64}{6}x^6 \quad (3.11)$$

According to the HPM, we can conclude that

$$y=\lim_{p\to 1} y(x)=y_0+\mathrm{p}y_1+\mathrm{p}^2y_2 \quad (3.12)$$

After putting the eqns. (3.10)-(3.11) into the eqn. (3.12) respectively, we obtain the solutions.

$$y(x)=4x+\frac{16}{3}x^3+8x^4+\frac{64}{6}x^6 \quad (3.13)$$

**Figure 1 (for example 2)**

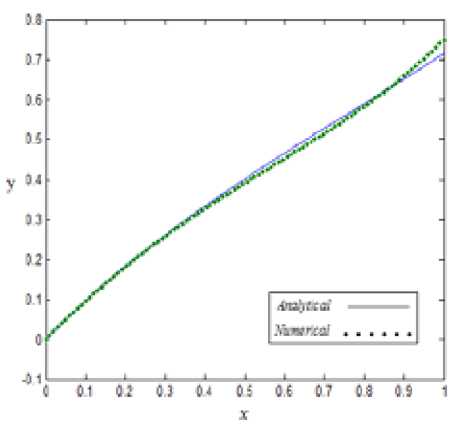
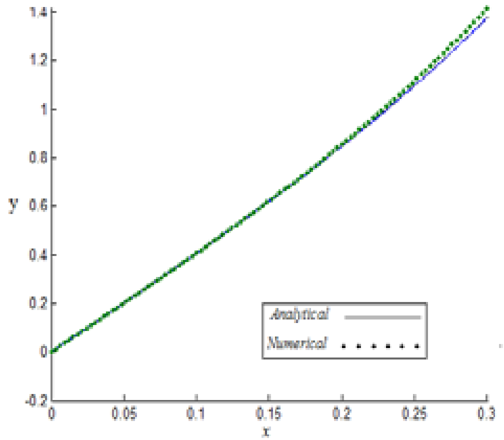



**Figure 2 (for example 3)**




**4. Numerical simulation**

The non-linear differential equations (2.1-2.2) and (3.1-3.2) are also solved by numerical methods using Matlab/Scilab software. Its numerical solution is compared with Homotopy perturbation method in Figs. (1) and (2) and it gives satisfactory result for the small rage of *x*. The Matlab/Scilab program is also given in Appendix (A).

**ERROR TABLE**

| Example 2(2.13) | | | | Example 3(3.13) | | | |
|---|---|---|---|---|---|---|---|
| $x$ | Numerical | Analytical | Error% | $x$ | Numerical | Analytical | Error% |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.1 | 0.9502 | 0.9502 | 0 | 0.05 | 0 | 0.2007 | 0 |
| 0.2 | 0.1804 | 0.1804 | 0 | 0.1 | 0 | 0.4061 | 0 |
| 0.3 | 0.2570 | 0.2570 | 0 | 0.15 | 0 | 0.6222 | 0 |
| 0.4 | 0.3264 | 0.3264 | 0 | 0.2 | 0 | 0.8561 | 0 |
| 0.5 | 0.3906 | 0.3906 | 0 | 0.25 | 0 | 1.117 | 0.0089 |
| 0.6 | 0.4524 | 0.4524 | 0 | 0.3 | 0 | 1.417 | 0.0706 |
| 0.7 | 0.5150 | 0.5150 | 0 | --- | --- | --- | --- |
| 0.8 | 0.5824 | 0.5824 | 0 | --- | --- | --- | --- |
| 0.9 | 0.6590 | 0.6590 | 0 | --- | --- | --- | --- |
| 1 | 0.7500 | 0.7500 | 0 | --- | --- | --- | --- |

**5. Discussion and Conclusion**

In example 1, we derived the exact solution of the system of first order ordinary linear differential equations. In example 2 and 3, we derived the approximate analytical solutions of the non-linear ordinary differential equations namely the Abelian differential equations. In this paper, He's Homotopy perturbation method has been successfully applied to find the solution of the system of linear and non-linear differential equations (Abelian differential equations) of the first order. The method is reliable and easy to use. The main advantage of the method is the fact that it provides its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed term. The HPM is an extremely simple compared to other method and it is also a promising method to solve other non-linear differential equations. This method can be easily extended to find the solution of all other strongly non-linear differential equations.

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