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Generalized Hyers - Ulam Stability of Quadratic Functional Equation in Paranormed Spaces

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ABSTRACT *In this paper, we obtain the general solution of a new generalized quadratic functional equation* $f(nx+y)+f(nx-y)=f(x+y)+f(x-y)+2(n^2-1)\,f(x)$ *in paranormed spaces for* $n \neq \pm 1$ *, n is an integer. Also we investigate the Hyers - Ulam stability of this functional equation.*

INTRODUCTION

The stability problem of functional equation originated from a question of S.M. Ulam [17]. In 1940, S. M. Ulam gave the following question concerning the stability of homomorphisms: *Under what Condition does there exist a homomorphism near an approximate homomorphism?*. In 1941, D.H. Hyers [9] answered the problem of Ulam under the assumption that the groups are Banach spaces. In 1950 T. Aoki [1] generalized the Hyers theorem for additive mappings. In 1978, Th.M. Rassias [12] provided a generalized version of the theorem of Hyers which permitted the Cauchy difference to become unbounded, where $f: X \to Y$ satisfies the inequality $\|f(x+y)-f(x)-f(y)\| \le \theta(\|x\|^p\,\|y\|^p)$ for all x, $y \in X$ for some $\theta \ge 0$ and $0 \le p < 1$. In 1982-1989, J.M. Rassias [13] gave a further generalization of the result of D.H. Hyers by proving the following theorem (1.1) using weaker conditions controlled by a product of different powers of norms.

Theorem 1.1. *Let* $f: E \to E'$ *be a mapping from a normed vector space E in to a Banach Space* E' *subject to the inequality* $\|f(x+y)-f(x)-f(y)\| \le \varepsilon \|x\|^p\,\|y\|^p$ *for all* $x \in E$ *times where* ε *and* p *are constants with* $\varepsilon > 0$ *and* $0 \le p < \frac{1}{2}$. *Then the limit* $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ *exists for all* $x \in E$ *and* $L: E \to E'$ *is the unique additive mapping which satisfies* $\|f(\mathrm{x})-\mathrm{L(x)}\| \le \frac{\varepsilon}{2-2^{2p}}\|x\|^{2p}$ *for all* $x \in E$. *Moreover, if f(tx) is continuous in t for each fixed* $x \in X$*, then the function L is linear.*

In 1990, Th.M.Rassias asked whether such a theorem can also be proved for $p \ge 1$. In 1991, Gajda [7] gave an affirmative solution to this question when $p > 1$, but it was proved by Gajda and Rassias and Semrl that one cannot prove an analogous theorem when $p = 1$. In 1994, a generalization was obtained by Gavruta [8] who replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. In 1996, Isac and Th.M. Rassias were the first to provide applications of stability theorem of functional equations.

A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [15] for mappings $f: X \to Y$ where X is a normed space and Y is a Banach Space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an abelian group. In 1992, Czerwik [5] proved the generalized Hyers-Ulam stability of the quadratic functional equation. In 2008, J.M. Rassias [13] introduced mixed type product-sum of powers of norms. Recently, Ch. Park [3] and D.Y. Shin [3] proved the Hyers-Ulam stability of the Cauchy additive, quadratic, cubic and the quartic functional equation in paranormed spaces. C. Park proved the Hyers-Ulam stability of an additive-quadratic-cubic-quartic functional equation

$$\begin{aligned} f(x+2y)+f(x-2y) &= 4f(x+y)+4f(x-y) \\ &\quad -6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y) \end{aligned}$$

(1.1) in paranormed spaces using fixed point method and direct method. K. Ravi, J.M. Rassias and B.V. Senthil Kumar [11] proved the Hyers-Ulam stability of the reciprocal difference functional equation

$$f\left(\frac{x+y}{2}\right) - f(x+y) = \frac{f(x)f(y)}{f(x)+f(y)} \qquad (1.2)$$

and adjoint functional equation

$$f\left(\frac{x+y}{2}\right) + f(x+y) = \frac{3f(x)f(y)}{f(x)+f(y)} \qquad (1.3)$$

Now we recall some basic facts concerning Frechet spaces. The concept of statistical convergence for sequences of real numbers was introduced by Fast[6] and Steinhaus[16] independently and since then several generalization and applications of this notion have been investigated by various authors. This notion was defined in normed spaces by E.Kolk.

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Definition 1.1: Let X be a vector space. A paranorm $P: X \to [0,\infty)$ is a function on X such that

(i) P(0)=0;

(ii) P(-x)=P(x);

(iii) P(x+y) ≤P(x)+P(y)(triangle inequality);

(iv) if $\{t_n\}$ *is a sequence of scalars with* $t_n \to t$ *and* $\{x_n\} \subset X$ *with* $P(x_n - x) \to 0$*, then* $P(t_n x_n - tx) \to 0$ *(continuity of multiplication).* The pair (X, P) is called a paranormed space, if P is a paranorm on X. The paranorm is total if, in addition, we have *(v) P(x)=0 implies x=0.* A Frechet space is a total and complete paranormed space. Throughout this paper, assume that (X, P) is a Frechet space and that $(Y, \|\cdot\|)$ is a Banach space.

In this paper, we investigate the generalized Hyers-Ulam stability of the functional equation

$$f(nx+y)+f(nx-y) = f(x+y)+f(x-y)+2(n^2-1)\ f(x) \qquad (1.4)$$

in paranormed spaces.

2. GENERAL SOLUTION

The following theorem provides the general solution of the functional equation (1.4) by establishing a connection with the classical quadratic functional equation.

Theorem 2.1. *Let X and Y be real vector spaces. A function* $f: X \to Y$ *satisfies the functional equation*

$$f(nx+y)+f(nx-y) = f(x+y)+f(x-y)+2(n^2-1)\ f(x) \qquad (2.1)$$

for all x, y ∈ X if and only if it satisfies the quadratic functional equation

$$f(x+y)+f(x-y) = 2f(x)+2f(y) \qquad (2.2)$$

for all x ∈X.

Proof. Suppose a function $f: X \to Y$ satisfies (2.1). Setting *(x, y) = (0, 0)* in (2.1), we obtain *f(0)=0.* Setting *(x, y) = (x, 0)* in (2.1), we obtain $2f(nx) = 2f(x)+ 2n^2f(x) - 2f(x)$ which gives $f(nx) = n^2f(x)$ for all $x \in X$ where *n* is a positive integer. Setting *(x, y) = (x, 2x)* in (2.1), we get *f(x) = f(-x)* for all *x, y*

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$\in X$. Setting $(x, y) = (x, x + y)$ in (2.1), we obtain

$$f((n+1)x+y)+f((n-1)x-y) = f(2x+y)+f(y)+2f(nx)-2f(x). \quad (2.3)$$

. Again, setting $(x,y)=(x,-y)$ in (2.3), we obtain

$$f((n+1)x-y)+f((n-1)x+y) = f(2x-y)+f(y)+2f(nx)-2f(x). \quad (2.4)$$

Adding (2.3) and (2.4). we obtain

$$f((n+1)x+y)+f((n-1)x+y)+f((n-1)x+y) +f((n-1)x-y)=f(2x+y)+f(2x-y)+2f(y) +4f(nx)-4f(x). \quad (2.5)$$

. Setting $n = n\text{-}1$ in (2.1), we obtain

$$f((n+1)x+y)+f((n-1)x-y) = f(x+y)+f(x-y)+2f((n-1)x) -2f(x). \quad (2.6)$$

Putting $n=2$ in (2.1), we get

$$f((2x+y)+f(2x-y) = f(x+y)+f(x-y)+2f(2x)-2f(x). \quad (2.7)$$

Substituting(2.6) and (2.7) in (2.5), we obtain

$$f((n+1)x+y)+f((n+1)x-y) =2f(y)+4f(nx)-4f(x)-2f((n-1)x) +2f(2x). \quad (2.8)$$

Putting $y = 0$ in (2.8), we get

$$2f((n+1)x)=4f(nx)-4f(x)-2f((n-1)x) +2f(2x). \quad (2.9)$$

Substituting(2.9) in (2.8), we obtain

$$f((n+1)x+y)+f((n+1)x-y) =2f(y)+2f((n+1)x). \quad (2.10)$$

Setting $((n + 1)x, y) = (x, y)$ in equation (2.10), we get

$$f(x+y)+f(x-y)=2f(x)+2f(y).$$

Conversely, suppose that a function $f: X \to Y$ satisfies (2.2). Putting $(x, y) = (0, 0)$ in equation (2.2), we obtain $f(0) = 0$. Setting $x = 0$ in (2.2) we obtain $f(-y) = f(y)$. Setting $(x, y) = (x, x)$ in equation (2.2), we obtain $f(2x) = 4f(x)$. Again setting $(x, y) = (x, 2x)$ in equation (2.2), we get $f(3x) = 9f(x)$. Setting $(x, y) = (x, (n-1)x)$ in equation (2.2), we obtain $f(nx) = n^2 f(x)$ for all positive integer n. Setting $(x, y) = (nx + y, nx - y)$ in equation (2.2), we get

$$f(nx+y+nx-y)+f(nx+y-nx+y) =2f(nx+y)+2f(nx-y) \quad (2.11)$$

which gives

$$f(nx+y)+f(nx-y) =\frac{1}{2}[f(2nx)+f(2y)] =2n^2 f(x)+2f(y)$$

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$$= 2n^2 f(x) - 2f(x) + 2f(x) + 2f(y)$$
$$= 2(n^2-1) f(x) + (2f(x) + 2f(y)) \quad (2.12)$$

Substituting (2.2) in (2.12), we obtain

$$f(nx+y) + f(nx-y) = f(x+y) + f(x-y) + 2(n^2-1)\, f(x)$$

3. GENEALIZED HYERS-ULAM STABILITY

The following theorem gives a general condition for which a true quadratic function exists near an approximately quadratic function. Let us denote

$$Df(x,y) = f(nx+y) + f(nx-y) - f(x+y) - f(x-y) - 2(n^2-1)\, f(x).$$

Theorem 3.1. *Let r be a positive real number with $r < 2$, and let $f : X \to Y$ be a mapping satisfying f(0)=0 and*

$$\left\| f(nx+y) + f(nx-y) - f(x+y) - f(x-y) - 2(n^2-1) f(x) \right\| \le P(x)^r + P(y)^r \quad (3.1)$$

for all x, y ∈ X. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\| f(x) - Q(x) \| \le \frac{1}{2(n^2 - n^r)} P(x)^r \quad (3.2)$$

for all x∈X.

Proof. Setting $y = 0$ in (3.1), we obtain

$$\left\| f(nx) - n^2 f(x) \right\| \le \frac{1}{2} P(x)^r .$$

Dividing the above inequality by n^2, we get

$$\left\| \frac{f(nx)}{n^2} - f(x) \right\| \le \frac{1}{2n^2} P(x)^r \quad (3.3)$$

for all $x \in X$.

Replacing x by nx and dividing by n^2, we obtain

$$\left\| \frac{f(n^2 x)}{n^4} - \frac{f(x)}{n^2} \right\| \le \frac{n^r}{2n^4} P(x)^r . \quad (3.4)$$

Adding(3.3) and (3.4), we obtain

$$\left\| \frac{f(n^2 x)}{n^4} - f(x) \right\| \le \frac{1}{2n^2} \left(1 + \frac{n^r}{n^2} \right) P(x)^r . \quad (3.5)$$

Generalizing, we get

$$\left\| \frac{f(n^m x)}{n^{2m}} - f(x) \right\| \le \frac{1}{2n^2} \sum_{k=0}^{m-1} \frac{n^{rk}}{n^{2k}} P(x)^r \quad (3.6)$$

for every positive integer *n, m* and for all $x \in X$. Replacing x by n^{sx} and dividing by n^2s in equation (3.6), we obtain

$$\frac{1}{n^{2s}} \left\| \frac{f(n^{m+s} x)}{n^{2m}} - f(n^s x) \right\| \le \frac{1}{2n^{s(2-r)+2}} \sum_{k=0}^{m-1} \frac{n^{rk}}{n^{2k}} P(x)^r . \quad (3.7)$$

By condition $r < 2$, the right hand side of (3.7) approaches 0 as $s \to \infty$ for all $x \in X$.

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Thus the sequence $\left\{\frac{f(n^m x)}{n^{2m}}\right\}$ is a cauchy sequence. Since X is complete, we can define a mapping $Q : X \to Y$ such that

$$Q(x) = \lim_{m\to\infty}\left\{\frac{f(n^m x)}{n^{2m}}\right\} \quad (3.8)$$

for all $x \in X$. Now we claim that the mapping $Q : X \to Y$ is a quadratic which satisfies the equation (2.1). Setting *(x, x)* = *($n^m x$, $n^m y$)* in equation (3.1) respectively and dividing by n^{2m}, we obtain

$$\left\|\frac{D(f(n^m x, n^m y))}{n^{2m}}\right\| \le \frac{1}{n^{2m}}\left(P(n^m x)^r + P(n^m y)^r\right) \quad (3.9)$$

It follows that from (3.1) that

$$\begin{aligned}&\left\|\begin{matrix}Q(nx+y)+Q(nx-y)-Q(x+y)\\ -Q(x-y)-2(n^2-1)Q(x)\end{matrix}\right\| \\ &= \lim_{m\to\infty}\frac{1}{n^{2m}}\left\|\begin{matrix}f(n^m(nx+y))+f(n^m(nx-y))\\ -f(n^m(x+y))-f(n^m(x-y))\\ -2(n^2-1)\ f(n^m(x))\end{matrix}\right\| \\ &\le \lim_{m\to\infty}\frac{1}{n^{m(2-r)}}\left(P(x)^r+P(y)^r\right) \\ &=0.\end{aligned}$$

which gives

$$Q(nx+y)+Q(nx-y) = Q(x+y)+Q(x-y) + 2(n^2-1)Q(x)$$

for all $x, y \in X$ and so the mapping $Q : X \to Y$ is quadratic. By taking the limit as $m \to \infty$ in equation (3.6), we obtain

$$\|f(x)-Q(x)\| \le \frac{1}{2(n^2-n^r)}P(x)^r$$

for all $x \in X$.

Now, let $T : X \to Y$ be another quadratic mapping satisfying the equation (3.2). Then we have

$$\begin{aligned}\|Q(x)-T(x)\| &\le \frac{1}{n^{2m}}\left(\begin{matrix}\|Q(n^m x)-f(n^m x)\|+\\ \|f(n^m x)-T(n^m x)\|\end{matrix}\right) \\ &\le \frac{1}{(n^2-n^r)n^{m(2-r)}}P(x)^r \quad (3.10)\end{aligned}$$

By condition (3.1), the right hand side of equation (3.9) approaches 0 as $m \to \infty$. We conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q.

Thus the mapping $Q : X \to Y$ is unique quadratic mapping satisfying (3.2).

Theorem 3.2. Let r, θ be positive real numbers with $r > 2$ and let $f : Y \to X$ be a mapping satisfying $f(0)=0$ and

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$$P\begin{pmatrix} f(nx+y)+f(nx-y)-f(x+y) \\ -f(x-y)-2(n^2-1)f(x) \end{pmatrix} \leq \theta\left(\|x\|^r+\|y\|^r\right) \quad (3.11)$$

for all $x, y \in Y$. Then there exists a unique quadratic mapping $Q: Y \to X$ such that

$$P\left(f(x)-Q(x)\right) \leq \frac{\theta}{2(n^r-n^2)}\|x\|^r \quad (3.12)$$

for all $x \in Y$.

Proof: Letting $y = 0$ in (3.1), we get

$$P\left(f(nx)-n^2 f(x)\right) \leq \frac{\theta}{2}\|x\|^r \quad (3.13)$$

for all $x \in Y$.

Dividing by n^2 on both sides,

$$P\left(\frac{1}{n^2}f(nx)-f(x)\right) \leq \frac{\theta}{2n^2}\|x\|^r \quad (3.14)$$

for all $x \in Y$.

Replacing x by $\frac{x}{n}$ and multiply both sides by n^2 in (3.14), we obtain

$$P\left(f(x)-n^2 f(\frac{x}{n})\right) \leq \frac{\theta}{2n^r}\|x\|^r \quad (3.15)$$

for all $x \in Y$.

Again Replacing x by $\frac{x}{n}$ and multiply both sides by n^2 in equation (3.15), we obtain

$$P\left(n^2 f(\frac{x}{n})-n^4 f(\frac{x}{n^2})\right) \leq \frac{\theta n^2}{2n^{2r}}\|x\|^r \quad (3.16)$$

Combining (3.15) and (3.16) we obtain,

$$P\left(f(x)-n^4 f(\frac{x}{n^2})\right) \leq \frac{\theta}{2n^r}\|x\|^r\left(1+\frac{n^2}{n^r}\right) \quad (3.17)$$

which can be extended by mathematical induction on m, we obtain

$$P\left(f(x)-n^{2m} f(\frac{x}{n^m})\right) \leq \frac{\theta}{2n^r}\sum_{k=0}^{m-1}\frac{n^{2k}}{n^{kr}}\|x\|^r \quad (3.18)$$

for every positive integer $m \geq 1$ and for all $x \in Y$. We have to show that the sequence $\left\{\frac{f(n^s x)}{n^{2s}}\right\}$ converges for all $x \in Y$. For every positive integer m and s, replacing x by $\frac{x}{n^s}$ and multiplying by n^{2s} on both sides in (3.18), we obtain

$$P\left(f(\frac{x}{n^s})n^{2s}-n^{2m+2s}f(\frac{x}{n^{m+s}})\right) \leq \frac{\theta}{2n^{s(r-2)}}\sum_{k=0}^{m-1}\frac{n^{2k}}{n^{r(k+1)}}\|x\|^r. \quad (3.19)$$

By condition (3.12), the right-hand side approaches 0 as $s \to \infty$ for all $x \in X$. Thus, the sequence $\left\{n^{2m} f\left(\frac{x}{n^m}\right)\right\}$ is a Cauchy sequence for all $x \in Y$, Since X is complete,

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the sequence $\left\{n^{2m} f\left(\frac{x}{n^m}\right)\right\}$ converges. So we can define the mapping $Q : Y \to X$ by

$$Q(x) = \lim_{m\to\infty}\left\{n^{2m} f\left(\frac{x}{n^m}\right)\right\} \tag{3.20}$$

for all $x \in Y$. Now we claim that the mapping $Q : Y \to X$ is a quadratic which satisfies the equation (3.11). Setting $(x, y) = (n^m x, n^m y)$ in equation (3.11) respectively and dividing by n^{2m}, we obtain

$$P\left(\frac{Df(n^m x, n^m y)}{n^{2m}}\right) \le \frac{1}{n^{2m}}\theta\left(\|n^m x\|^r + \|n^m y\|^r\right) \tag{3.21}$$

it follows from (3.11)that

$$P\begin{pmatrix} Q(nx+y)+Q(nx-y)-Q(x+y) \\ -Q(x-y)-2(n^2-1)Q(x)\end{pmatrix}$$
$$\le \lim_{m\to\infty} n^{2m} P\begin{pmatrix} f\left(\frac{nx+y}{n^m}\right)+f\left(\frac{nx-y}{n^m}\right)-f\left(\frac{x+y}{n^m}\right) \\ -f\left(\frac{x-y}{n^m}\right)-2(n^2-1)f\left(\frac{x}{n^m}\right)\end{pmatrix}$$
$$\le \lim_{m\to\infty} n^{2m}\,\theta\left(\left\|\frac{x}{n^m}\right\|^r + \left\|\frac{y}{n^m}\right\|^r\right)$$
$$\le \lim_{m\to\infty} \frac{\theta}{n^{m(r-2)}}\left(\|x\|^r + \|y\|^r\right)$$
$$=0.$$

for all $x, y \in Y$. Hence

$$Q(nx+y)+Q(nx-y) = Q(x+y)+Q(x-y) + 2(n^2-1)Q(x)$$

for all $x, y \in Y$ and so the mapping $Q : Y \to X$ is quadratic. By taking the limit as $m \to \infty$ in (3.19) and using (3.20), we obtain

$$P\left(f(x)-Q(x)\right) \le \frac{\theta}{2(n^r-n^2)}\|x\|^r.$$

Now let $T : Y \to X$ be another quadratic mapping satisfying (3.12), then we have

$$P\left(Q(x)-T(x)\right) = P\left(n^{2m}\left(Q\left(\frac{x}{n^m}\right)-T\left(\frac{x}{n^m}\right)\right)\right)$$
$$\le n^{2m} P\left(Q\left(\frac{x}{n^m}\right)-T\left(\frac{x}{n^m}\right)\right)$$
$$\le n^{2m}\begin{pmatrix} P\left(Q\left(\frac{x}{n^m}\right)-f\left(\frac{x}{n^m}\right)\right)+ \\ P\left(T\left(\frac{x}{n^m}\right)-f\left(\frac{x}{n^m}\right)\right)\end{pmatrix}$$
$$\le \frac{n^{2m}\theta}{(n^r-n^2)n^{mr}}\|x\|^r$$
$$\le \frac{\theta}{(n^r-n^2)n^{m(r-2)}}\|x\|^r$$

which tends to 0 as $m \to \infty$ for all $x \in Y$. So we can conclude $Q(x) = T(x)$ for all $x \in Y$. This proves the uniqueness of Q. Thus the mapping $Q : Y \to X$ is a unique quadratic mapping satisfies (3.12).

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