Linear Programming Problem with Homogeneous Constraints

KEYWORDS: Linear programming problem, Homogeneous constraints, Transformation matrix.

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ABSTRACT *This paper proposes an algorithm for solving a linear programming problem when some of its constraints are homogeneous. Using these homogeneous constraints a transformation matrix T is constructed. The matrix T transforms the given problem into another linear programming problem but with fewer constraints. A relationship between these two problems, which ensures that the solution of the original problem can be recovered from the solution of the transformed problem, is established. A simple numerical example illustrates the steps of the proposed algorithm.*

1. INTRODUCTION

Linear programming is a special type of problem in which all relations among the variables are linear , both in the constraints and the functions to be optimized. Under the assumptions that the set of the feasible solutions is a convex polyhedral with a finite number of extreme points. We have many iterative algorithms Charnes and Cooper(1962) [2], Martos (1960) [7], Swarup (1965)[8] to solve such problems . Usually, we can solve this problem through any one of the simplex method, the Big-M method and the two- Phase method[9].

The intention here is to reduce the computing time of the optimization process when a block of constraints are homogeneous. The method seems to be beneficial to large class of the linear programming models containing a great number of homogeneous constraints. Such constraints are encountered in transportation, flow and network models, Dantzing(1963) [3],Gass (1985) [4].

A transformation matrix T , which eliminates the homogeneous constraints. Section 2 describes the development of the transformation matrix and create a new algorithm. Section 3 presents the desired relationship between the original problem and the transformed problem. Numerical example and conclusion are given in the last two sections.

2. DEVELOPMENT OF THE TRANSFORMATION MATRIX

Let the given problem be to Maximize

$Z = CX \rightarrow$ (2.1)

Subject to $AX \leq b$ $\rightarrow$ (2.2)

and $X \geq 0$

with $a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots . + a_{ik}x_k + \cdots .. + a_{il}x_l + \cdots + a_{in}x_n = 0 \rightarrow$ (2.3)

$for\ some\ i$. Where $C = (\ c_{1,}c_2 \ldots c_n\ )$ is a row vector with n components.

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$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{pmatrix}$$ are column vectors and $A = (a_{ij})$;i=1,2,3…..m; j=1,2,3,…n

Where the constraint space is $L = \{X: AX = b; X \geq 0\}$

REMARK:

Let $X = (x_1, x_2, x_3, \dots x_n)$ be solution of (2.3). If x_k and $a_{ik} > 0$ then it is obvious that there exists at least one $x_l > 0$ with $a_{il} < 0$. In view of the remark, We partition matrix A as $A = (A^0, A^+, A^-)$ where A^0 is the set of all column of A whenever $a_{ij} = 0$. Let the number of such columns be r. A^+ is the set of all column of A whenever $a_{ij} > 0$. Let the number of such columns be p. A^- is the set of all column of A whenever $a_{ij} < 0$. Let the number of such columns be q. Thus $p + q + r = n$.

Now we define a transformation matrix T with n rows and $pq + r$ columns such that the ith equation of $ATW = b$ Will be identically zero. Here w is a column vector with $pq + r$ components. This is accomplished by defining variable w_{kl} for each pair (k,l) such that $A_k \epsilon A^+$ and $A_l \epsilon A^-$

Now partition T= $(T_1 : T_2)$ where T_1consist of unit column vectors e_j corresponding to $a_{ij} = 0$. T_2consist of pq column vectors corresponding to variables w_{kl}. The transformation matrix T can be represented asT= $(T_1 : T_2) = [(e_j), \forall j \epsilon a_{ij} = 0; (t_{kl}), \forall k \epsilon A^+; \forall l \epsilon A^-]$

That is e_j is the jth column of identity matrix I_n and $t_{kl} = -a_{il}\, e_k + a_{ik}\, e_l$. → (2.4)

From the above, we can create a new algorithm when the linear programming problem(L.P.P) must have homogeneous constraints.

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Step 1: Write the given L.P.P. in to its standard from.

Step 2: Select a homogeneous constraint from the given L.P.P.

Step 3: Find the number of positive terms, the number of negative terms and the number of zero terms in the selected homogeneous constraints. These are denoted by p,q and r respectively.

Step 4: From step (3), Find the order of identity matrix using the relation $p + q + r = n$.

Step 5: Construct the transformation matrix using the relation

T= $(T_1 : T_2) = [(e_j), \forall j \epsilon a_{ij} = 0; (t_{kl}), \forall k \epsilon A^+; \forall l \epsilon A^-]$

That is e_j is the j[th] column of identity matrix I_n and $t_{kl} = -a_{il}\ e_k + a_{ik} e_l$.

Step 6: Using the relation X =TW, the given problem can be transformed

Maximize $Z = CTW$, Subject to $A\ T$ W $= b$ and W ≥ 0

If there is no homogeneous constraints then go to step 7.If there is homogeneous constraints then go to step2. Continue this process until the homogeneous constraints are completely removed.

Step 7: Solve the transformed problem using by ordinary simplex method.

Step 8: The solution of the original problem obtained from the transformed problem using the relation X = TW.

2.TRANSFORMED PROBLEM AND RELATIONSHIP

Using the transformation matrix X = TW. We define the following problem.

Maximize $Z = \bar{C}$ W $\rightarrow$ (3.1), Subject to $\bar{A}$ W= b $\rightarrow$ (3.2)and W ≥ 0

Where $\bar{C} = CT$; $\bar{A} = AT$. Let $G = \{W : \bar{A}\,W = b; W \geq 0\}$ be constraint space.

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Theorem 3.1. If X solves (2.2) then there exist W which solves (3.2).

To prove this theorem, We need the following lemma.

Lemma 3.1. If

$\sum_{i=1}^{p} a_i = \sum_{j=1}^{q} \beta_j = V;\ a_i \geq 0;\ \beta_j \geq 0$

then there exist a matrix $Y = (y_{ij} \geq 0)$ such that $\sum_{j=1}^{q} y_{ij} = a_i$ and $\sum_{i=1}^{p} y_{ij} = \beta_j$

Proof of the lemma. The existence of such a matrix is ensured by $y_{ij} = \frac{a_i\ \beta_j}{V}$. For

$\sum_{i=1}^{p} y_{ij} = \frac{\beta_j\ \sum_{i=1}^{p} a_i}{V} = \beta_j$, For

$\sum_{j=1}^{q} y_{ij} = \frac{a_i\ \sum_{j=1}^{q} \beta_j}{V} = a_i$ and of course

y_{ij} as defined above ≥ 0.

Proof of the theorem.

As X solves (2.2) , its i[th] constraint equation will be $\sum_{k\epsilon A^+} a_{ik}\ x_k + \sum_{l\epsilon A^-} -a_{il}\ x_l = 0$.

Writing $a_{ik} x_k = a_k$ and $-a_{il} x_l = \beta_l \quad \rightarrow \quad (3.3)$

We have $\sum_{k=1}^{p} a_k = \sum_{l=1}^{q} \beta_l$. From the lemma 3.1, there exist a p by q matrix $Y = (y_{kl} \geq 0)$ such that

$\sum_{l=1}^{p} y_{kl} = a_k$ *and* $\sum_{k=1}^{q} y_{kl} = \beta_l$. Now we are ready to define vector W= $\binom{w^1}{w^2}$ where w^1 is a column vector with *r* components and w^2 is a column vector with *pq* components.

$w_j^1 x_j \ ; \forall j \epsilon A^0$ and

$w_{kl}^2 = \frac{-y_{kl}}{a_{ik} . a_{il}};\ \forall k \epsilon A^+; \forall l \epsilon A^- \quad \rightarrow \quad (3.4)$

Clearly W ≥ 0. Next we wish to show that $\bar{A}$ W= b , which amounts in showing that X =TW.

Consider TW = $T_1 w^1 + T_2 w^2$. Using (2.4) and (3.4).

This reduces to TW = $\sum_j e_j w_j^1 + \sum_k \sum_l t_{kl} w_{kl}^2 =$

$\sum_j e_j X_j + \sum_k \sum_l (-a_{il}\ e_k + a_{ik} e_l) \frac{-y_{kl}}{a_{ik} . a_{il}}$

TW $= \sum_j e_j X_j + \sum_k \frac{e_k\ a_k}{a_{ik}} - \sum_l \frac{e_l \beta_l}{a_{il}}$,

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In view of (3.3) We have,

$$TW = \sum_{j\epsilon A^0} e_j X_j + \sum_{k\epsilon A^+} e_k X_k + \sum_{l\epsilon A^-} e_l X_l$$

TW = X , This proves the theorem.

Theorem 3.2. If X^* solves the problem (2.1) – (2.2) then W^* solves the problem (3.1) – (3.2).

Proof: Theorem 3.1 guarantees the existence of a feasible W^*. That is

$\bar{A} W^* = b;\ W^* \geq 0$

$ATX^* = b;\ X^* \geq 0$

Next X^* solves the problem (2.1) - (2.2) imply that $C X^* \geq CX$; $\forall\ X\epsilon L$

This implies that $CTW^* \geq CTW$; $\forall\ W\epsilon G$

$\bar{C} W^* \geq \bar{C} W$; $\forall\ W\epsilon G$

Thus W^* solves the problem (3.1) – (3.2).

Theorem 3.3. If W^* solves the problem (3.1) – (3.2) then there exist $X^* = TW^*$ which solves the problem (2.1) – (2.2). Also the extreme values of the two objective functions are equal.

Proof:

W^* solves the problem (3.1) – (3.2) means that $\bar{A} W^* = b;\ W^* \geq 0$ or $ATW^* = b;\ X^* \geq 0$ or

$AX^* = b;\ X^* \geq 0 \quad \rightarrow \quad (3.5)$

Further , $T \geq 0 , W^* \geq 0$ implies that $X^* \geq 0 \quad \rightarrow \quad (3.6)$

Also we know that $\bar{C} W^* \geq \bar{C} W$; $\forall\ W\epsilon G \quad \rightarrow \quad (3.7)$

And would like to show that $C X^* \geq CX$; $\forall\ X\epsilon L \quad \rightarrow \quad (3.8)$

If possible , let $\bar{X}$ and not X^* solve the problem (3.1) – (3.2) which means that $C\bar{X} > C X^*$

From theorem (3.1) it follows that $CT\bar{W} > CT W^*$ or $\overline{CW} > \bar{C} W^*$

This violates (3.7) , the contradiction proves the result. Finally, let Z^* and z^* be the

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values of (2.1) and (3.1) at X^* and W^* respectively. This means $Z^* = C\,X^* = CTW^* = \bar{C}\,W^* = z^* \rightarrow$ (3.9)

The result then follows from

(3.5), (3.6), (3.8) *and* (3.9)

2.NUMERICAL EXAMPLE

In this section, we indicate how our new method is differ from the Two-phase method. First we solve a L.P.P. using Two- phase method. Consider an L.P.P.

Maximize $Z = 2x_1 + 6x_2$ Subject to $x_1 + x_2 + x_3 = 4$, $3x_1 + x_2 + x_4 = 6$

$x_1 - x_2 = 0$ and $x_1, x_2, x_3, x_4 \geq 0$.

Solution: The above problem can be written as Maximize $Z = 2x_1 + 6x_2 + 0x_3 + 0x_4 - 1x_5$

Subject to $x_1 + x_2 + x_3 + 0x_4 + 0x_5 = 4$, $3x_1 + x_2 + 0x_3 + x_4 + 0x_5 = 6$,

$x_1 - x_2 + 0x_3 + 0x_4 + 1x_5 = 0$ and

$x_1, x_2, x_3, x_4, x_5 \geq 0$.

Here x_3, x_4, x_5 are basic variables; x_1, x_2 are non- basic variables and x_5 is an artificial variable.

Phase – I: Assigning a cost -1 to the artificial variable and zero cost to all other variables . The objective function of the auxiliary L.P.P. becomes Maximize $Z^* = 0x_1 + 0x_2 + 0x_3 + 0x_4 - 1x_5$

Table:1. Initial Iteration

C_j : 0 0 0 0 -1

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	x_5	Ratio
0	x_3	4	1	1	1	0	0	4
0	x_4	6	3	1	0	1	0	2
-1	x_5	0	**1**	-1	0	0	1	0

$Z_j - C_j$ -1 1 0 0 0

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Table :2. First Iteration: Here x_5 drops from the basis and x_1 enters the basis.

Cj : 0 0 0
0 -1

<table>
<tr><td>C_B</td><td>Y_B</td><td>X_B</td><td>x_1 x_2 x_3
x_4 x_5</td></tr>
<tr><td>0

0

0</td><td>x_3

x_4

x_1</td><td>4

6

0</td><td>0 2 1
0 -1
0 4 0
1 -3 **1**
-1 0 0
1</td></tr>
</table>

Z_j- C_j 0 0
0 0 1

Here all Z_j- $C_j \geq 0$ and the artificial variable does not lie in the basis. So we go to Phase –II.

Phase –II We consider the actual cost associated with original variables. The new objective function is Maximize Z = $2x_1 + 6 x_2 + 0x_3 + 0 x_4$.

Table :3. Initial Iteration :

C_j : 2
6 0 0

<table>
<tr><td>C_B</td><td>Y_B</td><td>X_B</td><td>x_1
x_2
x_3
x_4</td><td>Ratio</td></tr>
<tr><td>0
0
2</td><td>x_3
x_4
x_1</td><td>4
6
0</td><td>0
2
1
0
0
4
0
1
1
-1
0
0</td><td>2
3/2
-</td></tr>
</table>

Z_j- C_j 0
-8 0 0

Tabel:4 . Final Iteration: In this iteration x_4 drops and x_2 enters in the basis.

Cj : 2
6 0 0

<table>
<tr><td>C_B</td><td>Y_B</td><td>X_B</td><td>x_1
x_2 x_3
x_4</td></tr>
<tr><td>0
6
2</td><td>x_3
x_2
x_1</td><td>1
3/2
3/2</td><td>0
0 1
-1/2
0
1 0
1/4
1
0 0
1/4</td></tr>
</table>

Z_j- C_j 0
0 0 2

Since all $Z_j - C_j \geq 0$, We reached optimum solution .

Therefore Maximum Z = 12when $x_1 = x_2 = 3/2$, $x_3 = 1$ and $x_4 = 0$.

This method have totally four Iterations. The same problem , in the B- M have three iterations. But our new method have two iterations only. Our new method illustrated through the same problem as follows. The problem can be written as in the matrix form

Maximize $Z = 2x_1 + 6x_2$; Here, Let

$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix}$; $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ and

$b = \begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix}$

Here $x_1 - x_2 + 0x_3 + 0x_4 + 1x_5 = 0$ is a homogeneous constraint. According to the new algorithm $p = 1, q = 1$ and $r = 2$. Now $n = 4$ this implies $I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and

$t_{kl} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.Therefore $T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

For the transformation problem, $CTW =$

$\begin{pmatrix} 2 & 6 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 8w_3$.

$ATW = b$ implies

$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} =$

$\begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix}$

This implies $w_1 + 2w_3 = 4$ and $w_2 + 4w_3 = 6$

Thus the given problem can be transformed as Maximize $Z = 8w_3$,Subject to $w_1 + 2w_3 = 4$, $w_2 + 4w_3 = 6$ and $w_1, w_2, w_3 \geq 0$

This transformed problem have two constraints only and the homogeneous constraint is completed eliminated. Solution of the transformed problem is as follows

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Tabel:5. Initial iteration:

Cj : 0 0 8

C_B	Y_B	w_B	w_1 w_2 w_3	ratio
0	W_1	4	1 0 2	2
0	W_2	6	0 1 **4**	1.5

Z_j- C_j 0 0 -8

Since ,there is one $Z_j - C_j < 0$.

Therefore ,we go to next iteration.

Table:6. First iteration: W_3 enters to the basis and W_2 drops from the basis.

C_j : 0 0 8

C_B	Y_B	W_B	w_1 w_2 w_3
0	W_1	1	1 -1/2 0
8	W_3	3/2	0 1/4 1

Z_j- C_j 0 2 0

Since all $Z_j - C_j \geq 0$. Therefore we reached optimum solution. That is Maximum Z = 12

when w_1 = 1and w_3=3/2.Now we get solution of the original problem using the relation X = TW.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3/2 \end{pmatrix}$$

$$= \begin{pmatrix} 3/2 \\ 3/2 \\ 1 \\ 0 \end{pmatrix}$$

Here the same solution reached as Maximum Z = 12 when $x_1 = x_2 = 3/2$, x_3=1 and x_4= 0 .

But total number of iterations reduced as well as computing time reduced.

2.CONCLUSION

The process described in section 2, can be extended to define T if AX = b has more than one homogeneous constrains. In case there are s homogeneous constraints, we define s transformation matrices T(1), T(2),

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T(3), ….T(s). T(2) is determined once AT(1) has been computed. In general T(s) is determined only when AT(1) ,AT(2) ….AT(s-1) has been computed. This algorithm which reduces the number of constraints the main factor of the optimization complexity, can be used efficiently for solving large- scale programming problems.

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