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COLOS RODING	Linear Programming Problem with Homogeneous Constraints				
KEYWORDS	Linear programming problem, Homogeneous constraints, Transformation matrix.				
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are homogeneous. Using these homogeneous constraints a transformation matrix T is constructed. The matrix T transforms the given problem into another linear programming problem but with fewer constraints. A relationship between these two problems, which ensures that the solution of the original problem can be recovered from the solution of the transformed problem, is established. A simple numerical example illustrates the steps of the proposed algorithm.

1. INTRODUCTION

Linear programming is a special type of problem in which all relations among the variables are linear, both in the constraints and the functions to be optimized. Under the assumptions that the set of the feasible solutions is a convex polyhedral with a finite number of extreme points. We have many iterative algorithms Charnes and

Cooper(1962) [2], Martos (1960) [7],

Swarup (1965)[8] to solve such problems. Usually, we can solve this problem through any one of the simplex method, the Big-M method and the two- Phase method[9].

The intention here is to reduce the computing time of the optimization process when a block of constraints are homogeneous. The method seems to be beneficial to large class of the linear programming models containing a great number of homogeneous constraints. Such constraints are encountered in transportation, flow and network models,

Dantzing(1963) [3],Gass (1985) [4].

A transformation matrix T , which eliminates the homogeneous constraints. Section 2 describes the development of the transformation matrix and create a new algorithm. Section 3 presents the desired relationship between the original problem and the transformed problem. Numerical example and conclusion are given in the last two sections.

2. DEVELOPMENT OF THE TRANSFORMATION MATRIX

Let the given problem be to Maximize
$Z = CX \rightarrow (2.1)$
Subject to $AX \le b \rightarrow (2.2)$
and $X \ge 0$
with $a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 +$
$\cdots + a_{ik}x_k + \cdots + a_{il}x_l + \cdots + a_{in}x_n =$
$0 \to (2.3)$
for some <i>i</i> . Where $C = (c_1, c_2 \dots c_n)$ is a
row vector with n components.

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \quad , \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ b_m \end{pmatrix} \text{ are column}$$

vectors and $A = (a_{ij})$; i=1,2,3....m; j=1,2,3,...n

Where the constraint space is L =

 $\{X: AX = b; X \ge 0\}$

REMARK:

Let $X = (x_{1}, x_{2}, x_{3}, ..., x_{n})$ be solution of (2.3). If x_{k} and $a_{ik} > 0$ then it is obvious that there exists at least one $x_{l} > 0$ with $a_{il} < 0$. In view of the remark, We partition matrix A as $A = (A^{0}, A^{+}, A^{-})$ where A^{0} is the set of all column of A whenever $a_{ij} = 0$. Let the number of such columns be r. A^{+} is the set of all column of Awhenever $a_{ij} > 0$. Let the number of such columns be p. A^{-} is the set of all column of A whenever $a_{ij} < 0$. Let the number of such columns be q. Thus p + q + r = n. Now we define a transformation matrix T with n rows and pq + r columns such that the ith equation of ATW = b Will be identically zero. Here w is a column vector with pq + r components. This is accomplished by defining variable w_{k1} for each pair (k,1) such that $A_k \epsilon A^+$ and $A_l \epsilon A^-$

Now partition $T = (T_1: T_2)$ where T_1 consist of unit column vectors e_j corresponding to $a_{ij} = 0$. T_2 consist of pq column vectors corresponding to variables w_{kl} . The transformation matrix T can be represented as $T = (T_1: T_2) = [(e_j), \forall j \in a_{ij} =$ 0; $(t_{kl}), \forall k \in A^+; \forall l \in A^-]$

That is e_j is the jth column of identity matrix I_n and $t_{kl} = -a_{il} e_k + a_{ik} e_l$. \rightarrow (2.4)

From the above, we can create a new algorithm when the linear programming problem(L.P.P) must have homogeneous constraints. **Step 1**: Write the given L.P.P. in to its standard from.

Step 2: Select a homogeneous constraint from the given L.P.P.

Step 3: Find the number of positive terms, the number of negative terms and the number of zero terms in the selected homogeneous constraints. These are denoted by p,q and r respectively.

Step 4: From step (3), Find the order of identity matrix using the relation p + q + r = n.

Step 5: Construct the transformation matrix using the relation

 $T = (T_1: T_2) = [(e_j), \forall j \in a_{ij} = 0; (t_{kl}), \forall k \in A^+; \forall l \in A^-]$

That is e_j is the jth column of identity matrix

 $I_n \text{ and } t_{kl} = -a_{il} e_k + a_{ik} e_l.$

Step 6: Using the relation X =TW, the given problem can be transformed

Volume : 4 | Issue : 3 | Mar 2014 | ISSN - 2249-555X Maximize Z = CTW, Subject to ATW= b and $W \ge 0$

If there is no homogeneous constraints then go to step 7. If there is homogeneous constraints then go to step2. Continue this process until the homogeneous constraints are completely removed.

Step 7: Solve the transformed problem using by ordinary simplex method.

Step 8: The solution of the original problem obtained from the transformed problem using the relation X = TW.

2.TRANSFORMED PROBLEM AND RELATIONSHIP

Using the transformation matrix X = TW. We define the following problem.

Maximize $Z = \overline{C} W \rightarrow (3.1)$, Subject

to $\overline{A} W = b \rightarrow (3.2)$ and $W \ge 0$

Where $\overline{C} = CT$; $\overline{A} = AT$. Let $G = \{W: \overline{A} | W = b; W \ge 0\}$ be constraint space.

Theorem 3.1. If X solves (2.2) then there

exist W which solves (3.2).

To prove this theorem, We need the following lemma.

Lemma 3.1. If

 $\sum_{i=1}^{p} a_i = \sum_{j=1}^{q} \beta_j = V; \ a_i \ge 0; \ \beta_j \ge 0$ then there exist a matrix $Y = (y_{ij} \ge 0)$ such that $\sum_{j=1}^{q} y_{ij} = a_i$ and $\sum_{i=1}^{p} y_{ij} = \beta_j$

Proof of the lemma. The existence of such a matrix is ensured by $y_{ij} = \frac{a_i \beta_j}{V}$. For

 $\sum_{i=1}^{p} y_{ij} = \frac{\beta_j \sum_{i=1}^{p} a_i}{V} = \beta_j \text{, For}$ $\sum_{j=1}^{q} y_{ij} = \frac{a_i \sum_{j=1}^{q} \beta_j}{V} = a_i \text{ and of course}$

 y_{ij} as defined above ≥ 0 .

Proof of the theorem.

As X solves (2.2), its ith constraint equation will be $\sum_{k \in A^+} a_{ik} x_k^+$

 $\sum_{l\in A^-} -a_{il} x_l = 0.$

Writing $a_{ik}x_k = a_k$ and $-a_{il}x_l =$

 $\beta_l \rightarrow (3.3)$

We have $\sum_{k=1}^{p} a_k = \sum_{l=1}^{q} \beta_l$. From the lemma 3.1, there exist a p by q matrix $Y = (y_{kl} \ge 0)$ such that $\sum_{l=1}^{p} y_{kl} = a_k$ and $\sum_{k=1}^{q} y_{kl} = \beta_l$. Now we are ready to define vector $W = {\binom{w^1}{w^2}}$ where w^1 is a column vector with *r* components and w^2 is a column vector with *pq* components.

$$w_j^1 x_j$$
; $\forall j \in A^0$ and
 $w_{kl}^2 = \frac{-y_{kl}}{a_{ik} \cdot a_{il}}; \quad \forall k \in A^+; \forall l \in A^- \rightarrow (3.4)$

Clearly $W \ge 0$. Next we wish to show that $\overline{A} W = b$, which amounts in showing that X =TW.

Consider $TW = T_1w^1 + T_2w^2$. Using (2.4) and (3.4).

This reduces to TW = $\sum_{j} e_{j} w_{j}^{1} +$

 $\sum_{k} \sum_{l} t_{kl} w_{kl}^{2} =$ $\sum_{j} e_{j} X_{j} + \sum_{k} \sum_{l} (-a_{il} e_{k} + a_{ik} e_{l}) \frac{-y_{kl}}{a_{ik} \cdot a_{il}}$

$$TW = \sum_{j} e_{j} X_{j} + \sum_{k} \frac{e_{k} a_{k}}{a_{ik}} - \sum_{l} \frac{e_{l\beta_{l}}}{a_{il}} ,$$

In view of (3.3) We have,

$$TW = \sum_{j \in A^0} e_j X_j + \sum_{k \in A^+} e_k X_k +$$
$$\sum_{l \in A^-} e_l X_l$$

TW = X, This proves the theorem.

Theorem 3.2. If X^* solves the problem (2.1) - (2.2) then W^* solves the problem (3.1) - (3.2).

Proof: Theorem 3.1 guarantees the existence of a feasible W^* . That is $\overline{A} W^* = b; W^* \ge 0$

 $ATX^* = b; X^* \ge 0$

Next X^* solves the problem (2.1) - (2.2) imply that $C X^* \ge CX$; $\forall X \in L$

This implies that $CTW^* \ge CTW$; $\forall W \in G$

 $\overline{C} W^* \ge \overline{C} W ; \forall W \epsilon G$

Thus W^* solves the problem (3.1) - (3.2).

Theorem 3.3. If W^* solves the problem (3.1) - (3.2) then there exist $X^* = TW^*$

which solves the problem (2.1) - (2.2).

Also the extreme values of the two objective functions are equal.

Proof:

 W^* solves the problem (3.1) – (3.2) means that $\overline{A} W^* = b$; $W^* \ge 0$ or $ATW^* = b$; $X^* > 0$ or $AX^* = b; X^* \ge 0 \longrightarrow (3.5)$ Further, $T \ge 0$, $W^* \ge 0$ implies that X^* $\geq 0 \rightarrow (3.6)$ Also we know that $\overline{C} W^* \ge \overline{C} W$; $\forall W \epsilon G$ (3.7)And would like to show that $CX^* \ge CX$; $\forall X \in L \rightarrow (3.8)$ If possible, let \overline{X} and not X^* solve the problem (3.1) – (3.2) which means that $C\bar{X}$ $> C X^*$ From theorem (3.1) it follows that $CT\overline{W}$ $> CT W^*$ or $\overline{CW} > \overline{C} W^*$ This violates (3.7), the contradiction proves

the result. Finally, let Z^* and z^* be the

values of (2.1) and (3.1) at X^{*} and W^{*} respectively. This means $Z^* = C X^* = CTW^*$ $= \overline{C} W^* = z^* \rightarrow (3.9)$

The result then follows from

(3.5), (3.6), (3.8) and (3.9)

2.NUMERICAL EXAMPLE

In this section, we indicate how our new method is differ from the Two-phase method. First we solve a L.P.P. using Two- phase method. Consider an L.P.P.

Maximize $Z = 2x_1 + 6x_2$ Subject to x_1 + $x_2 + x_3 = 4$, $3x_1 + x_2 + x_4 = 6$

 $x_1 - x_2 = 0$ and $x_1, x_2, x_3, x_4 \ge 0$.

Solution: The above problem can be written as Maximize $Z = 2x_1 + 6x_2 + 0 x_3 + 0x_4 - 1x_5$

Subject to $x_1 + x_2 + x_3 + 0x_4 + 0x_5 = 4$, 3 $x_1 + x_2 + 0x_3 + x_4 + 0x_5 = 6$,

$$x_1 - x_2 + 0x_3 + 0x_4 + 1x_5 = 0$$
 and

 $x_1, x_2, x_3, x_{4}, x_5 \ge 0$.

Here x_3 , x_4 , x_5 are basic variables; x_1 , x_2 are non-basic variables and x_5 is an artificial variable.

Phase – I:Assigning a cost -1 to the artificial variable and zero cost to all other variables . The objective function of the auxiliary L.P.P. becomes Maximize $Z^* =$ $0x_1 + 0 x_2 + 0 x_3 + 0 x_4 - 1x_5$

Table:1. Initial Iteration

Cj: 0 0 0

C _B	Y _B	X _B	X ₁ X ₃ X ₅	X4	X2	Ratio
0 0 -1	X ₃ X ₄ X ₅	4 6 0	1 1 3 0	0	1 0 1 0	4 2 0
Zj- 0	C _j 0		1 0 -1 0	0	-1 1	

Table :2. First Iteration: Here x_5 drops from the basis and x_1 enters the basis.

		Cj :	0		0	0
0		-1				
C _B	Y _B	X _B	x ₁ x ₄	X 5	x ₂	X3
0	X3	4	0		2	1
0	X4	6	0	-1		
0	x ₁	0	0		4	0
			1	-3	1	
			-1	0	0	
			1			
Z _i - (- Li	1	1	0		0

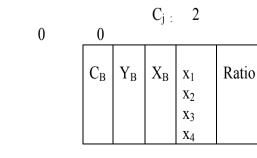
 $2_j - C_j = 0$ 0 = 0 1

Here all Z_j - $C_j \ge 0$ and the artificial variable does not lie in the basis. So we go to Phase –II.

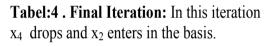
Phase –II We consider the actual cost associated with original variables. The new objective function is Maximize $Z = 2x_1 + 6$ $x_2 + 0x_3 + 0 x_4$.

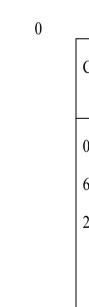
Table :3. Initial Iteration :

6



	0	X3	4	0	2
				2	
	0	X4	6	2 1 0	3/2
	2		0	0	
	2	X ₁	0	0	-
				0	
				4	
				4 0 1	
				1	
				1	
				1 -1 0	
				0	
				0	
		Z _j - C _j		0	
0	()			





0

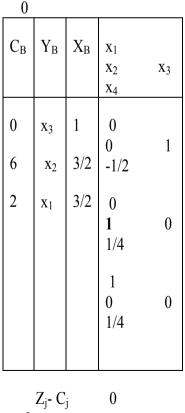
2

-8

6

0

 $Cj\ :\ 2$



Since all Z_j - $C_j \ge 0$, We reached optimum solution .

Therefore Maximum Z = 12 when $x_1 = x_2 = 3/2$, $x_3 = 1$ and $x_4 = 0$.

This method have totally four Iterations. The same problem , in the B- M have three iterations. But our new method have two iterations only. Our new method illustrated through the same problem as follows. The problem can be written as in the matrix form

Maximize $Z = 2x_1 + 6x_2$; Here, Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix}; \ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ and}$$
$$b = \begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix}$$

Here $x_1 - x_2 + 0x_3 + 0x_4 + 1x_5 = 0$ is a homogeneous constraint. According to the new algorithm p = 1, q = 1 and r = 2. Now *n*

= 4 this implies
$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
 and

$$t_{kl} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$
. Therefore $T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

For the transformation problem, CTW =

$$(2 \quad 6 \quad 0 \quad 0) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 8w_3.$$

ATW = b implies

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix}$$

This implies $w_1 + 2w_3 = 4$ and $w_2 + 4w_3 = 6$ Thus the given problem can be transformed as Maximize $Z = 8w_3$, Subject to $w_1 + 2w_3 = 4$, $w_2 + 4w_3 = 6$ and w_1 , w_2 , $w_3 \ge 0$

This transformed problem have two constraints only and the homogeneous constraint is completed eliminated. Solution of the transformed problem is as follows Tabel: 5. Initial iteration:

	Cj :	0	-	0	8	
	C _B	Y _B	WB	W1 W2 W3	ratio	
	0 0	W ₁ W ₂	4	1 0 2 0 1 4	2 1.5	
Zj- Cj		0		0		-8

Since , there is one $Z_j - C_j < 0$.

Therefore ,we go to next iteration.

Table:6. First iteration: W_3 enters to the basis and W_2 drops from the basis.

$$C_{1:} 0 0 8$$

2

0

Since all Z_j - $C_j \ge 0$. Therefore we reached optimum solution. That is Maximum Z = 12

when $w_1 = 1$ and $w_3 = 3/2$. Now we get solution of the original problem using the relation X = TW.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3/2 \end{pmatrix}$$
$$= \begin{pmatrix} 3/2 \\ 3/2 \\ 1 \\ 0 \end{pmatrix}$$

Here the same solution reached as Maximum Z = 12 when $x_1 = x_2 = 3/2$, $x_3=1$ and $x_4=0$.

But total number of iterations reduced as well as computing time reduced.

2.CONCLUSION

The process described in section 2, can be extended to define T if AX = b has more than one homogeneous constrains. In case there are s homogeneous constraints, we define s transformation matrices T(1), T(2), T(3),T(s). T(2) is determined once AT(1) has been computed. In general T(s) is determined only when AT(1) ,AT(2)AT(s-1) has been computed. This algorithm which reduces the number of constraints the main factor of the optimization complexity, can be used efficiently for solving large- scale programming problems.

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