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Subjective Probability and Expected Utility, A Stochastic Approximation Evaluation

KEYWORDS: preferences, subjective probability, utility, stochastic approximation, non-additive measure.

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ABSTRACT *The topic of this article is stochastic algorithms for evaluation of the utility and subjective probability based on the decision maker's preferences. The main direction of the presentation is toward development of mathematically grounded algorithms for subjective probability and expected utility evaluation as a function of both the probability and the rank of the alternative. The stochastic assessment is based on mathematically formulated axiomatic principles and stochastic procedures and on the utility theory without additivity.*

The uncertainty of the human preferences is eliminated as is typical for the stochastic programming. Numerical presentations are shown and discussed.

INTRODUCTION

The representation of complex systems including human decisions as objective function needs mathematical tools for evaluation of qualitative human knowledge. In decision making theory the primitive are preferences relations as description of people's strategies, guided both by internal expectations about their own capabilities of getting results, and by external feedback of this result (Keeney & Raiffa, 1993). Such modeling addresses theory of measurement (scaling), utility theory and Bayesian approach in decision making. The Bayesian statistical technique in decision making is applicable when the information and uncertainty in respect of problems, hypothesis and parameters can be expressed by probability distribution and functional representation of human preferences (Griffiths & Tenenbaum, 2006). Such an approach needs careful analysis of the terms *measurement*, formalization and admissible *mathematical operations* in the modeling. This is a fundamental level that requires the use of basic mathematical terms like sets, relations and operations over them, and their gradual elaboration to more complex and specific terms like value and utility functions, operators on mathematically structured sets and harmonization of these descriptions with set of axioms. In this aspect we enter the theory of measurements and the expected utility theory (Fishburn, 1970).

The evaluation of qualitative human knowledge and the mathematical inclusion of the subjective probabilities and utility posed many difficulties and needs a special attention. Generally the human notions and preferences have qualitative or verbal expression. The wisely merge of the qualitative and verbal expression as human preferences and quantitative mathematical description causes many efforts. The violations of transitivity of the preferences lead to declinations in utility and subjective probability assessment (Cohen & al., 1988; Kahneman & Tversky, 1979; Fishburn, 1988; Machina, 2009). Such declinations explain the DM behavior observed in the Allais Paradox (Allais, 1953). A long discussion for the role of the mathematic and the Bayesian theory in the human decision making reality has been started yet. New extensions of axiomatic bases of the developed mathematical theories are considered for further wide developments of von Neumann's theory. Fruitful directions of researches are development of a non-additive subjective utility theory. The mathematical results of Schmeidler in respect of subjective probability and utility description make a great impression on this development (Shmeidler, 1989).

The paper suggests a reasonable well-founded mathematical approach and methods for subjective probability and utility evaluation based on the von Neumann's utility theory and the Kahneman's and Schmeidler's findings. We propose and discuss a stochastic programming for subjective probability and utility polynomial evaluation as machine learning based on the human preferences. Numerical presentations are shown and discussed.

MATHEMATICAL FORMULATIONS AND BACKGROUND

The difficulties that come from the mathematical approach are due to the probability and subjective uncertainty of the DM expression and the cardinal character of the expressed human's preferences. The mathematical description is the following. Let $\boldsymbol{X}$ be the set of alternatives ($\boldsymbol{X} \subseteq \boldsymbol{R}^m$). From practical point of view the empirical system of human preferences relations is a algebraic system with relations SR ($\boldsymbol{X}$,($\approx$),($\succ$)), where ($\approx$) can be considered as the relation "indifferent or equivalent", and ($\succ$) is the relation "prefer". We look for equivalency of the empirical system with the numbered system of relations SR ($\boldsymbol{R}$-real numbers, (=), (>)). The *"indifference"* relation ($\approx$) is based on ($\succ$) and is defined by $((x \approx y) \neg((x \succ y) \vee (x \prec y)))$.

We introduce a set $\boldsymbol{S}$, which elements are named *state of nature,* following Schmaidler's exposition (Shmeidler, 1989). Let $\boldsymbol{\Omega}$ be algebra of subset of $\boldsymbol{S}$. Denote by $\boldsymbol{D_o}$ the set of all measurable finite step valued functions from $\boldsymbol{S}$ to $\boldsymbol{P}$ and denote by $\boldsymbol{D_c}$ the constant functions in $\boldsymbol{D_o}$. Let $\boldsymbol{D}$ be a convex subset of $\boldsymbol{P^{\Omega}}$ which includes $\boldsymbol{D_c}$, ($\boldsymbol{D_c} \subseteq \boldsymbol{D_o} \subseteq \boldsymbol{D}$). In the neo-Bayesian nomenclature elements of $\boldsymbol{X}$ are deterministic finite outcome (alternatives), elements of $\boldsymbol{P}$ are random outcomes or *lotteries* connected with the objective probabilities, and elements of $\boldsymbol{D}$ are *acts* connected with the uncertainty of human operations described with subjective probabilities. Elements of $\boldsymbol{S}$

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are *state of nature* and elements of $\boldsymbol{\Omega}$ are *events*. The relations ($\approx$) and ($\succ$) are primitives in the empirical system with relations SR($\boldsymbol{D}$, ($\approx$),($\succ$)). A *value function* is a function ($u^*: \boldsymbol{X} \to \boldsymbol{R}$) for which it is fulfilled:

$((x, y) \in \boldsymbol{X}^2, x \succ y) \Leftrightarrow (u^*(x) > u^*(y))$.

The assumption of existence of a value function $u(.)$ leads to the "*negatively transitive*" and "*asymmetric*" relation ($\succ$) - "*weak order*". A "*strong order*" is a "*weak order*" for which is fulfilled $(\neg(x \approx y) \Rightarrow ((x \succ y) \vee (x \prec y))$. The existence of a "*weak order*" ($\succ$) over $\boldsymbol{X}$ leads to the existence of a "*strong order*" over $\boldsymbol{X}/\approx$. Consequently the existence of a value function u(.) leads to the existence of: asymmetry $((x \prec y) \Rightarrow (\neg(x \succ y))$, transitivity $((x \succ y) \wedge (y \succ z) \Rightarrow (x \succ z))$ and transitivity of the "indifference" relation ($\approx$) (Fishburn, 1970).

The definition of *utility function* is more complex. Let $\boldsymbol{X}$ be a *set of alternatives* and $\boldsymbol{P}$ is a set of *probability distributions* over $\boldsymbol{X}$ and that $\boldsymbol{X} \subseteq \boldsymbol{P}$ ($x \in \boldsymbol{X} \Rightarrow \exists\, y \in \boldsymbol{P}$, $y(x)=1$). A utility function $u(.)$ will be any function for which is fulfilled:

$(p \succ q,\ (p,q) \in \boldsymbol{P}^2) \Leftrightarrow (\int u(.)dp > \int u(.)dq)$.

The relations ($\approx$) and ($\succ$) are primitives in the algebraic system with relations SR ($\boldsymbol{P}$,($\approx$),($\succ$)). To every choice corresponds a discrete probability distribution p, $p \in \boldsymbol{P}$ with finite domain of appearance of final results (alternatives). There are different systems of mathematical axioms that give satisfactory conditions of a utility function existence. The most famous of them is the system of Von Neumann and Morgenstern's axioms:

(**A.1**) *Preferences* relations ($\succ$) and ($\approx$) defined over $\boldsymbol{P}$ are transitive, i.e. the binary preference relation ($\succ$) is *weak order*;

(**A.2**) *Archimedean Axiom*: for all $p, q, r \in \boldsymbol{P}$ such that $(p \succ q \succ r)$, there is an $\alpha, \beta \in (0,1)$ such that $((\alpha p + (1-\alpha)r) \succ q)$ and $(q \succ (\beta p + (1-\beta)r))$;

(**A.3**) *Independence Axiom*: for all $p, q, r \in \boldsymbol{P}$ and any $\alpha \in (0, 1)$, then $(p \succ q)$ if and only if $((\alpha p + (1-\alpha)r) \succ (\alpha q + (1-\alpha)r))$.

Axioms (A1) and (A3) cannot give solution (Fishburn, 1970). Axioms (A1), (A2) and (A3) give solution in the interval scale (precision up to an affine transformation):

$((p \succ q) \Leftrightarrow (\int v(x)dp \succ \int v(x)dq) \Leftrightarrow (v(x)= \boldsymbol{a}u(x)+\boldsymbol{b},\ \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}, \boldsymbol{a}>0, x \in \boldsymbol{X}))$.

The assumption of existence of a utility (value) function $u(.)$ leads to the "*negatively transitive*" and "*asymmetric*" relation ($\succ$) and to transitivity of the relation ($\approx$). So far we are in the preference scale, the *ordering scale*. The assumption of equivalence with precision up to affine transformation has not been included yet. In other words we have only a value function. For value, however, the mathematical expectation is unfeasible, but we underline that the mathematical expectation is included in the definition of the utility function. For this reason it is accepted that ($\boldsymbol{X} \subseteq \boldsymbol{P}$) and that the set of objective probability distributions $\boldsymbol{P}$ is a convex set: $((q,p) \in \boldsymbol{P}^2 \Rightarrow (\alpha q+(1-\alpha)p) \in \boldsymbol{P}$, for $\forall \alpha \in [0,1])$. Then by the von Neuman – Morgenstern theorem the utility $u(.)$ is determined in the interval scale (Fishburn, 1970):

Theorem1: Let $\boldsymbol{P}$ be a convex set of finite distributions over the set $\boldsymbol{X}$ of final results and let $\boldsymbol{X}$ is included in $\boldsymbol{P}$, $\boldsymbol{X} \subseteq \boldsymbol{P}$. The axioms (A1), (A2) and (A3) are necessary and sufficient condition for existence of an affine real valued utility function over the convex set**.** This is means that if $((x \in X \wedge p(x)=1) \Rightarrow p \in \boldsymbol{P})$ and $(((q, p) \in \boldsymbol{P}^2) \Rightarrow ((\alpha p+(1-\alpha)q) \in \boldsymbol{P}, \alpha \in [0,1]))$ are realized, then the utility function $u(.)$ is defined with precision up to an affine transformation: $(u_1(x) \approx u_2(x), x \in \boldsymbol{X}) \Leftrightarrow (u_1(.)=au_2(.)+b, a>0)$.

Following this theorem, the measurement of the preferences defined over $\boldsymbol{P}$ is in the *interval scale*. That is to say, this is a utility function. Now it is obvious why in practice the gambling approach is used to construct the utility function in the sense of von Neumann (Keeney & Raiffa, 1993). The reason is that to be in the interval scale the set of the discrete probability distributions $\boldsymbol{P}$ have to be convex. The same holds true in respect of the set $\boldsymbol{X}$. The utility function is evaluated by the "gambling approach". This approach consists within the comparisons between lotteries. A "lottery" is called every discrete

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probability distribution over X. We denote as <x, y, α> the simplest lottery: α is the probability of the appearance of the alternative *x* and (1-α) - the probability of the alternative *y*. The weak points of the gambling approach are the violations of the transitivity of the preferences, the so called "certainty effect" and "probability distortion" (Cohen and al., 1988). Violations of the transitivity of the equivalence relation (≈) is a general case and also lead to declinations in the utility assessment. All these difficulties explain the DM behavior observed in the Allais Paradox. Schmeidler introduced four new axioms defined over SR (***D***, (≈), ($\succsim$)) (Schmeidler, 1989):

(**B.4**) *Comonotonic Independence*: For all pairwise *comonotonic* acts *f*, *g* and *h* in ***D*** and for all $\alpha\in(0, 1)$, ($f \succsim g$) implies ($\alpha f+\alpha h \succsim \alpha g + \alpha h$):

$(f \succsim g \Rightarrow \alpha f+ \alpha h \succsim \alpha g + \alpha h)$.

Two acts *f* and *g* in ***D*** are said to be *commonotonic* if for no *s* and *t* in ***S***, ($f(s) \succsim f(t)$) and ($g(t) \succsim g(s)$). It is obvious that a constant act *f* ($f = p^s$) for some *p*, $p\in \boldsymbol{P}$ and any constant act *g*, are *comonotonic*. Comonotonic independence is less restrictive then the independendence axiom.

(**B.5**) *Monotonicity*: For all acts *f*, *g* in ***D***: if for all *s*, $s\in \boldsymbol{S}$, ($f(s) \succsim g(s)$, $\succsim$ defined over ***P***) then ($f \succsim g$, ($\succsim$) defined over ***D***).

(**B.6**) *Strict Monotonicity*: For all acts *f* and *g* in ***D***, *p* and *q* in ***P*** and *E*, $E\in\boldsymbol{\Omega}$: if ($f \succsim g$), $f = p$ on *E* and $g = q$ on *E*, and ($f(s) = g(s)$) on E^c, then for the constant acts is true ($p^s \succsim q^s$).

(**B.7**) *Nondegeneracy*: There are *f* and *h* in ***D*** that $f \succsim h$.

When subjective probability enters into calculation of the expected utility of an act of ***D***, an integral with respect to a finite additive set function has to be defined. Denote by L a finitely additive probability measure on***Ω***, the algebra of subset of ***S*** and let *k*(.) be a real value ***Ω***–measurable function on ***S***. For the case where *k*(.) is a finite step function, can be represented by $\kappa = \sum_{i=1}^{H} \mathbf{a_i E_i^*}$, where ($a_1 > a_2 > ... a_i >,..> a_n$) are the values that *k*(.) attains and $\mathbf{E_i^*}$ is the indicator function of $E_i \cong \{s\in \boldsymbol{S} / k(s) = a_i\}$, i=1,..,n. By definition we enter the integral: $\int_S \kappa dL = \sum_{i=1}^{n} L(E_i)a_i$.

The Anscombe-Auman theorem is true (Fishburn, 1970; Schmeidler, 1989):

Theorem2: Suppose that a preference relation ($\succsim$) defined over $\boldsymbol{D_o}$ satisfies (A.1) Weak order, (A.3) Independence, (A.2) Archimedean axiom-continuity, (B.6) Strict Monotonicity and (B.7) Nondegeneracy. Then there exists a unique finitely additive probability measure L on ***Ω*** and an affine real valued utility function *u*(.) on ***X*** ($\int u(.)dp > \int u(.)dq$), $p \succsim q$, $p\in\boldsymbol{P}$, $q\in\boldsymbol{P}$) such that for all *f* and *g* in $\boldsymbol{D_o}$:

$$f \succ g \text{ if } \int_S u(f(.))dL > \int_S u(g(.))dL .$$

The utility function *u*(.) is defined over ***P***, $\boldsymbol{X\subseteq P}$, with precision up to an affine transformation:

$(u_1(x) \approx u_2(x), x \in \boldsymbol{X}) \Leftrightarrow (u_1(.)=au_2(.)+b, a>0)$.

A real valued set function *v*(.) is termed non-additive probability if it satisfies the normalization conditions $v(\varnothing)=0$ and $v(\boldsymbol{S})=1$. The function satisfies moniotonicity, i.e. for E and G in***Ω***: ($E \subseteq G$) implies ($v(E) \le v(G)$). Schmeidler introduces the following definition of ($\int_S k(x)dv$) for *v*(.) non-additive probability and $\kappa = \sum_{i=1}^{H} \mathbf{a_i E_i^*}$,where ($a_1 > a_2 > .. a_i >$,., $> a_n$) are the values that *k*(.) attains and $\mathbf{E_i^*}$ is the indicator function on of $E_i \cong \{ s\in\boldsymbol{S} / k(s) = a_i\}$, for i=1,..,n. Let $a_{n+1}=0$ and define:

$$\int_S \kappa dv = \sum_{i=1}^{n} (a_i - a_{i+1}) v(\bigcup_{j=1}^{i} E_j) .$$

This is the definition of the Choquet integral for finite step functions. For the case of non-additive subjective probability the following theorem is true (Schmeidler, 1989):

Theorem3: Suppose that the preference relation ($\succsim$) defined over $\boldsymbol{D_o}$ satisfies (A.1) - Weak order, (B.4) - Comonotonic independence, (A.2) - Archimedean axiom-continuity, (B.5) - Monotonicity, and (B.7) - Nondegeneracy. Then there exists a unique non-additive probability measure v on $\boldsymbol{\Omega}$ and an affine real valued utility function $u(.)$ on $\boldsymbol{X}$ ($\int u(.)dp > \int u(.)dq$, $p \succ q$, $p \in \boldsymbol{P}$, $q \in \boldsymbol{P}$) such that for all f and g in $\boldsymbol{D_o}$:

$$f \succ g \text{ if } \int_S u(f(.))\mathrm{d}v > \int_S u(g(.))\mathrm{d}v .$$

The utility function $u(.)$ is defined over $\boldsymbol{P}$, $\boldsymbol{X} \subseteq \boldsymbol{P}$ with precision up to an affine transformation:

$$(u_1(x) \approx u_2(x),\ x \in \boldsymbol{X}) \Leftrightarrow (u_1(.) = au_2(.) + b,\ a>0).$$

In Schmeidler's paper is proved an extended theorem for the more extended set $\boldsymbol{D}$ as a subset of $\boldsymbol{P}^{\boldsymbol{\Omega}}$ (acts connected with the uncertainty of the human operations described mathematically with subjective probabilities). Interesting for us is the case of $\boldsymbol{D_o}$ - all measurable finite step valued functions from $\boldsymbol{S}$ to $\boldsymbol{P}$ since in the practice the set of acts is a finite set.

Proceeding from these theorems and following the research of Kahneman, Tversky and the debates about the well known Allais paradox and similar paradoxes extensions and further developments of von Neumann's theory were sought. Among these theories the rank dependent utility (RDU) and its derivative cumulative Prospect theory are currently the most popular (Kahneman & Tversky, 1979). In the RDU the decision weight of an outcome is not just the probability associated with this outcome. It is a function of both the probability and the rank the alternative. For example, the RDU of the lottery (p_1, x_1; p_2, x_2; ...; p_n, x_n) is:

$$RDU = \sum_{i=1}^{n} W(p_i)u(x_i) .$$

Based on empirical researches several authors have argued that the probability weighting function W(.) has an inverse S-shaped form, which starts on concave and then becomes convex. It is supposed that $W(p_i) = p_i + \Delta W(p_i)$. The declination of the probability assessment, the probability distortion $\Delta W(p_i)$ has a S-shaped form closed to symmetry. The theoretical findings and the discussions described in short above give a hint for investigations of the process preferences evaluation as a function of both subjective probability (in the lottery) and rank of the alternative (utility) (1988; Kahneman, 1979; Schmeidler, 1989; Machina, 2009).

STOCHASTIC UTILITY EVALUATION

The theorems in the previous section oriented us to analytical evaluations as preferences function of two variables $u(.,.)$ - Utility of the alternative and the appropriate probability. It is proposed the following stochastic approximation procedure for evaluation of the utility function of two variables $u(.,.)$, named in short utility function. In correspondence with the theorem1, theorem2 and theorem3 it is assumed that $(\boldsymbol{X} \subseteq \boldsymbol{P})$, $((q, p) \in \boldsymbol{P}^2 \Rightarrow (\alpha q + (1-\alpha)p) \in \boldsymbol{P}$, for $\forall \alpha \in [0,1])$ and that the utility function $u(.,.)$ exists. The "lotteries" are discrete probability distribution over $\boldsymbol{X}$. Once again we denote as <x, y,α> the simplest lottery: α is the probability of the appearance of the alternative x and (1-α) - the probability of the alternative y. *The DM compares the "lottery" <x, y,α> with the simple alternative z,* $z \in \boldsymbol{X}$ and preferences are expressed as learning points for the machine learning stochastic procedure. We define two sets base on the form of the lotteries <x, y, α> and on the form $u(.,.)$ of the utility function:

$$\boldsymbol{A_{u^*}} = \{(x,y,z,\alpha) / (\alpha u^*(x,\alpha) + (1-\alpha)u^*(y, 1-\alpha)) > u^*(z,1)\},$$

$$\boldsymbol{B_{u^*}} = \{(x,y,z,\alpha) / (\alpha u^*(x,\alpha) + (1-\alpha)u^*(y,1-\alpha)) > u^*(z,1)\},$$
$\alpha \in [0,1]$.

The notation $u^*(.,.)$ is preserved for the DM's empirical human preferences and assessment. The following proposition is in the foundation of the used stochastic approximation approach (Pavlov & Andreev, 2013):

Proposition4: We denote $\boldsymbol{A_u} = \{(x, y, z, \beta, \gamma, \alpha) / (\alpha u(x, \beta) + (1-\alpha)u(y, \gamma)) > u(z,1)\}$, here $\alpha, \beta, \gamma \in [0,1]$. If $\boldsymbol{A_{u1}} = \boldsymbol{A_{u2}}$ and $u_1(x, \beta)$ and $u_2(x, \beta)$ are continuous functions than is true $(u_1(x, \beta) = au_2(x, \beta) + b,\ a>0)$.

The approximation of the utility function is based on pattern recognition of the set $\boldsymbol{A_u}$ []. The process is machine-learning based on the DM's preferences expressed in the framework of the gambling approach. This is a stochastic pattern recognition because ($A_{u^*}\cap B_{u^*}\neq\emptyset$). The utility evaluation is a stochastic approximation with noise elimination**.** The evaluation procedure is:

*The DM compares the "lottery" <x, y, α> with the simple alternative z, z∈**Z ("better-≻, f(x, y, z, α)=1", "worse-≺, f(x, y, z, α)=(-1)" or "can't answer or equivalent- ~ , f(x, y, z, α)=0", f(.) denotes the qualitative DM's preference and determines the answer).** This determine a learning point **((x, y, z, β, γ,α), f(x, y, z, α).** The following recurrent stochastic algorithm constructs the utility polynomial approximation* $u(x,\beta)=\sum_{i,j} c_{ij}\Phi_i(x)\Phi_j(\beta)$:

$$c_{ij}^{n+1}=c_{ij}^{n}+\gamma_n\left[f(t^{n+1})-\overline{(c^n,\Psi(t^{n+1}))}\right]\Psi_{ij}(t^{n+1}),$$

$$\sum_n \gamma_n=+\infty,\ \sum_n \gamma_n^2<+\infty,\ \forall n,\gamma_n>0.$$

In the formula are used the following notations (based on $\mathbf{A_u}$): $\mathbf{t}=(x, y, z, \beta, \gamma, \alpha)$, $\psi_{ij}(t)=\psi_{ij}(x, y, z, \beta, \gamma, \alpha)=\alpha\Phi_i(x)\Phi_j(\beta)+(1-\alpha)\Phi_i(y)\Phi_j(\gamma)-\Phi_i(z)\Phi_j(1)$, where $(\Phi_i(x))$ is a family of polynomials (possibly orthogonal). The line above the scalar product $\bar{v}=\overline{(c^n,\Psi(t))}$ means: $(\bar{v}=1)$, if (v>1), $(\bar{v}=-1)$ if (v<-1) and $(\bar{v}=v)$ if (-1<v<1). The coefficients c_{ij}^{n} take part in the polynomial presentation $g^n(x,\beta)=\sum_{i=1,j=1}^{n} c_{ij}^n\Phi_i(x)\Phi_j(\beta)$. The notation $G^n(x,y,z,\beta,\gamma,\alpha)=(c^n,\Psi(t))$, $(c^n,\Psi(t))=\alpha g^n(x,\beta)+(1-\alpha)g^n(y,\gamma)-g^n(z,1)$ is a scalar product. The learning points (x, β) are set with a pseudo random sequence.

The mathematical procedure describes the following assessment process: the expert relates intuitively the "learning point" (x, y, z, β, γ, α) (comparison of the "lottery" <*x, y, α*> with the alternative z) to the set $\mathbf{A_{u^*}}$ with probability $D_1(x, y, z, \alpha)$ or to the set $\mathbf{B_{u^*}}$ with probability $D_2(x, y, z, \alpha)$. The probabilities $D_1(x, y, z, \alpha)$ and $D_2(x, y, z, \alpha,)$ are mathematical expectation of $f(.)$ over $\mathbf{A_{u^*}}$ and $\mathbf{B_{u^*}}$ respectively, ($D_1(x, y, z, \alpha)=M(f/x, y, z, \alpha)$) if ($M(f/x, y, z, \alpha)>0$), ($D_2(x, y, z, \alpha)=(-)M(f/x, y, z, \alpha)$) if ($M(f/x$, $D'(x, y, z, \alpha)=D_1(x, y, z, \alpha, \beta)$ if ($M(f/x, y, z, \alpha)>0$); $D'(x, y, z, \alpha)=(D_2(x, y, z, \alpha)$ if ($M(f/x, y, z, \alpha)<0$); $D'(x, y, z, \alpha)=0$ if ($M(f/x, y, z, \alpha)=0$). We approximate function $D'(x, y, z, \alpha)$ by the function: $G(x, y, z, \beta, \gamma,\alpha)=(\alpha g(x, \beta)+(1-\alpha)g(y, \gamma)-g(z, 1))$, where $g(x,\beta)=\sum_{i=1,j=1}^{n} c_{ij}\Phi_i(x)\Phi_j(\beta)$. The coefficients c_{ij}^{n} take part in:

$$G^n(x,y,z,\beta,\gamma,\alpha)=(c^n,\Psi(t))=$$
$$=\alpha g^n(x,\beta)+(1-\alpha)g^n(y,\gamma)-g^n(z,1),$$

$$g^n(x,\beta)=\sum_{j=1,i=1}^{N} c_i^n\Phi_i(x)\Phi_j(\beta),$$

approximation of $G(x, y, z, \beta, \gamma,\alpha)$:

The function $G^n(x, y, z, \beta, \gamma, \alpha)$ is positive over $\mathbf{A_{u^*}}$ and negative over $\mathbf{B_{u^*}}$ depending on the degree of approximation of $D'(x, y, z, \alpha)$. In keeping with proposition 4 function $g^n(x, \beta)$ is the approximation of the utility function $u(x, \beta)$. Another form of the stochastic recurrent procedure is the following (Aizerman and al., 1970):

$$c_{ij}^{n+1}=c_{ij}^{n}+\gamma_n\left[D'(t^{n+1})+\xi^{n+1}-\overline{(c^n,\Psi(t^{n+1}))}\right]\Psi_{ij}(t^{n+1}),$$

$$\sum_n \gamma_n=+\infty,\ \sum_n \gamma_n^2<+\infty,\ \forall n,\gamma_n>0.$$

In the formula above is used the decomposition:

$$f(t^{n+1})=\left[D'(t^{n+1})+\xi^{n+1}\right].$$

The following theorem determines the convergence of the procedure (Pavlov & Andreev, 2013):

Theorem5: Let (t^1, t^2, t^3,......., t^n,.....) be a sequence of independent random vectors of the form $t=(x, y, z, \beta, \gamma, \alpha)$ with one and the same distribution F. We suppose that the sequence of random values (ξ^1, ξ^2,., ξ^n,..) in the recurrent stochastic procedure satisfies the conditions: $M(\xi^n/(x, y, z, \beta, \gamma, \alpha),c^{n-1})=0$, $M((\xi^n)^2/(x, y, z, \beta, \gamma, \alpha),c^{n-1})<d$, $d\in R$. Let the Euclidian norm of $\Psi(t)$ is limited by a constant,

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$$\left\|\Psi(t)\right\| < \theta, \theta \in \boldsymbol{R}, \theta > 0, \text{ for } \forall\, t, t = (x, y, z, \beta, \gamma, \alpha).$$

The following convergence is result from the stochastic evaluation procedure:

$$J_{D'}(G^n(x,y,z,\beta,\gamma,\alpha)) = M(\int_{D'(t)}^{G^n(t)} (\bar{v} - D'(t))dv) =$$

$$= \int (\int_{D'(t)}^{G^n(t)} (\bar{v} - D'(t))dv)dF \xrightarrow[n]{e.w.}$$

$$\xrightarrow[n]{e.w.} \inf_{S(t)} \int (\int_{D'(t)}^{S(t)} (\bar{v} - D'(t))dv)dF \; .$$

In the theorem above e. w. denotes “almost sure” (with probability 1) and M denotes mathematical expectation. The functions $S(t)$ in the limits of the integral belong to L_2 (the space of square integrable functions defined by the probability measure F). This function has the presentation:

$S(x, y, z, \beta, \gamma, \alpha)=(\alpha s(x, \beta)+(1-\alpha)s(y, \gamma)-s(z, 1))$.

The integral $J_{D'}(G^n(x, y, z, \alpha))$ fulfills (Pavlov & Andreev, 2013):

$$\int (\int_{D'(t)}^{G^n(t)} (\bar{v} - D'(t))dv)dF \geq \frac{1}{2} \int \overline{(G^n(t) - D'(t))^2} dF$$

The proof is based on the “extremal approach” of the Potential function method (Pavlov & Andreev, 2013). The procedure and its modifications are machine learning (Kivinen and al, 2004; Aizerman and al., 1970). The computer is taught to have the same preferences as the decision maker (DM). The learning points $((x, y, z, \alpha, \beta, \gamma), f(x, y, z, \alpha))$ are set with a pseudo random sequence. The DM is comparatively fast in learning to operate with the procedure: a session with 512 questions (learning points) takes approximately 90 minutes and requires only qualitative answers “yes”, “no” or “equivalent ”.

NUMERIC PRESENTATION AND VARIFICATIONS

The numerical verifications give again proof of the efficacy of proposed stochastic approximation approach. The proposed stochastic approach permits sufficiently precise utility function presentation by two variables function $u(.,.)$ of the DM’s preferences. Example of such an evaluated utility function $f_2(y,\alpha)$ is shown on figure 1. The seesaw surface $f_2(y,\alpha)$ in figure (1) is constructed by 512 DM’s “learning points” of the form $((x, y, z, \alpha, \beta, \gamma), f(x, y, z, \alpha))$, $f(x, y, z, \alpha) \in \{1, -1, 0\}$ (probabilistic pattern recognition of the sets $\boldsymbol{A}_{u*}$ and $\boldsymbol{B}_{u*}$, $\boldsymbol{A}_{u*} \cap \boldsymbol{B}_{u*} \neq \varnothing$). The explicit formula of the cumulative utility function $f_2(y)$ shown in figure 2 is:

$$f_2(y) = \int_0^1 f_2(y, \alpha) d\alpha \, .$$

The utility function is evaluated additionally and independently with 512 DM’s “learning points” $((x, y, z, \alpha), f(x, y, z, \alpha))$, $f(x, y, z, \alpha) \in \{1, -1, 0\}$. The probabilistic pattern recognition function $f_3(y)$ is shown on figure (2).

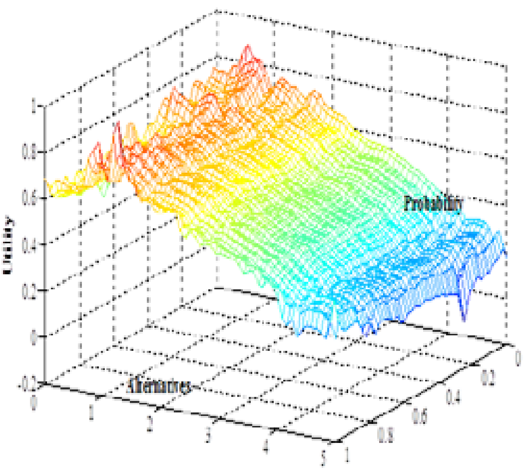

Figure 1: pattern recognition - $f_2(y, \alpha)$

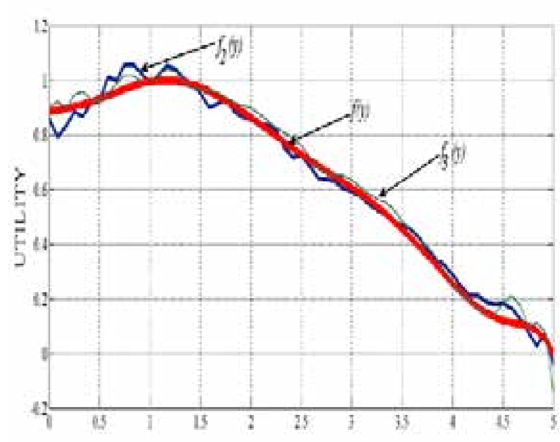

Figure 2: functions $f(y)$, $f_2(y)$, $f_3(y)$

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This is pattern recognition of the sets A_{u*} and B_{u*} without assuming dependence of utility from the probability in the lotteries and recognize correctly more than 95% of the DM's answers. The polynomial approximation of the utility is the function f(y) shown as the solid line on figure (2). This example shows that the procedure permits direct assessment of the utility function $f_2(y, \alpha)$ as a function of both probability (of the lottery) and the rank (alternative) y. If the probability weighting function W(.) in RDU utility has a symmetric form then is true:

$$\int_0^1 \Delta W(p) f_2(y) dp = 0.$$

In this case the expected von Neuman and Morgenstern's utility function $f_2(.)$ is exactly the integral because p is evenly distributed in the pseudo-random sequence:

$$\int_0^1 W(p) f_2(y) dp = \frac{1}{2} u(y) \ .$$

Thus, we could first evaluate $f_2(y, \alpha)$, following Kaneman, Tversky and Schmeidler theoretical findings and after that we could apply integration and the "certainty effect" and "probability distortion" could be reduced.

DISCUTIONS

The two variables utility function $u(x, \alpha)$ permits evaluation of the dependence of the utility on probability. The function has the following presentation: $u(x, \alpha) = (1+\Delta W(\alpha, x))u^*(x)$. This presentation permits calculation of an approximation of the Choquet integral:

$$\int_S \kappa(s) dv = \sum_{i=1}^{n} (a_i - a_{i+1}) v(\bigcup_{j=1}^{i} E_j) \ .$$

In the cases of discrete finite distribution (finite number of acts, finite probability distribution over the set of alternatives) the non-additive function v(p) is a function of the expression $(1 + \Delta W(p, x))$ where the function $\Delta W(p, x)$ is a part of the *RDU*. *RDU* could have in new notation the presentation:

$$RDU = (p + p\Delta W(p, x)) u^*(x).$$

The stochastic approach proposed in the paper gives a possibility for evaluation and approximations of the non-additive probability measure discussed in the Schmeidler's theorem 3 (Schmeidler, 1989).

CONCLUSIONS

The stochastic approach proposed in the paper is a possible approach for investigation and determination of new mathematical procedures for approximations of the non-additive probability measure discussed in theorem 3.

The numerical experiments and the mathematical modeling confirm a 95% precision of the approximations. The human uncertainty is found predominated to the dependence of the utility on probability in the experiments with the lotteries comparisons. The elimination of the uncertainty effect through a two variables function utility evaluation needs sufficiently high number of learning points in the machine learning (512 or 1024 points). In some cases as in the trading of a set of various articles the experiments show growing up of the utility dependence on probability. In such experiments the merge of different articles in the grocery is preferable in regard to the salesman and reflects the gain or loss of money as result of this variety of articles. In the rest of experiments the utility dependence on probability did not made real appearance in the process of utility evaluation based on the gambling approach.

The precision of the approximations could be increased by utilization of comparisons (preferences) between more complex lotteries in the gambling approach. Such a more complex lotteries will put wide requirements in regard to the decision maker.

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