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Exact Solution of Energy Fractional Equation

KEYWORDS: Energy equation, Fractional derivatives, Laplace transform, Fourier sine transform, Caputo fractional derivative

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ABSTRACT *Fourier sine transform and Laplace transform are used for solving the energy equation with fractional derivative, where the fractional derivative is defined in the Caputo sense of order* $m-1<\gamma\le m$ *. The solution of classical problem for the energy equation has been obtained as limiting case.*

1. Introduction

Fractional partial differential equations have many applications in applied sciences and engineering. These applications appear in gravitation elastic membrane, electrostatics, fluid flow, steady state, heat conduction and many other topics in both pure and applied mathematics. Typical examples partial differential equations of the time fractional advection dispersion equation as in[6,7], fractional diffusion equation as in[16,8,5,9,15], fractional wave equation as in[14]. The Rayleigh-stokes fractional equations as in[2].

The energy fractional equations are examples of fractional partial differential equation .

In this paper we consider energy fractional equation. Exact solution of this equation will be investigated. The Fourier sine transform and fractional Laplace transform are used for getting exact solution for this equation. The fractional terms in energy equation are considered as Caputo fractional derivative.

Basic Definitions:

Definition1: The Rieman-Liouville fractional integral[10,2] of order α is defined as:

$$J^{\alpha} f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\,dt \quad (1.1)$$

Definition 2: The Caputo fractional derivative [10] of order $m-1<\alpha\le m$ is defined as:

$$D^{\alpha} f(x)=\frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-m+1}}\,dt \quad (1.2)$$

Definition 3: The Laplace integral transform[11,13,4,10], of the function $f(x)$ is defined as:

$$L(f(x))=\int_0^{\infty} f(x)\,e^{-st}\,dx \quad (1.3)$$

Definition 4: The Fourier sine integral transform[4,10,1], of the function $f(x)$ is defined as:

$$F_e(f(x))=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\sin(\zeta x) f(x)dx \quad (1.4)$$

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2. The Energy Fractional Equation:

The time-fractional energy equation, when the Fourier's law of heat conduction is considered may be written in the form

$$\frac{k}{c\rho}\frac{\partial^2\theta(x,t)}{\partial x^2}+\frac{\nu}{c}\left[\frac{\partial u(x,t)}{\partial x}\right]^2 + \frac{r(x,t)}{\rho c}=\frac{\partial^\gamma\theta(x,t)}{\partial t^\gamma} \quad (2.1)$$

where r(x,t) is the radiant heating, which is neglected in this study, c is the specific heat and k is the conductivity which is assumed to be constant and $m-1<\gamma\leq m$. The corresponding initial and boundary conditions of Eq. (2.1) are

$$\frac{\partial^n\theta(x,0)}{\partial x^n}=a_n(x), \quad for \quad x>0 \quad and\ n\geq 0 \quad (2.2)$$

$$\theta(0,t)=T_0, \quad for \quad t\geq 0 \quad (2.3)$$

Moreover, the natural conditions

$$\theta(x,t), \frac{\partial^n\theta(x,t)}{\partial x^n}\to 0, \quad for \quad x\to\infty \ and\ n>0 \quad (2.4)$$

also have to be satisfied.

Applying the non-dimensional quantities

$$\theta^*=\frac{\theta}{T_0}, \nu*=\frac{u}{U}, x^*=\frac{xu}{\nu}, \quad t^*=\frac{tu^2}{\nu}, \lambda=\frac{U^2}{cT_0}, \mathrm{Pr}=\frac{c\mu}{k} \quad (2.5)$$

Eqs. (2.2), (2.3) and (2.4) can be reduce to non-dimensional equations as follows

$$\frac{1}{\mathrm{Pr}}\frac{\partial^2\theta(x,t)}{\partial x^2}+\lambda\left[\frac{\partial u(x,t)}{\partial x}\right]^2 =\frac{\partial^\gamma\theta(x,t)}{\partial t^\gamma} \quad (2.6)$$

$$\frac{\partial^n\theta(x,0)}{\partial t^n}=a_n(x), \quad for \quad x>0 \quad and\ n\geq 0 \quad (2.7)$$

$$\theta(0,t)=1, \quad for \quad t\geq 0 \quad (2.8)$$

$$\theta(x,t), \frac{\partial^n\theta(x,t)}{\partial x^n}\to 0, \quad for \quad x\to\infty \ and\ n>0 \quad (2.9)$$

Letting $g(x,t)=\lambda\left[\frac{\partial u(x,t)}{\partial x}\right]^2$, then Eq. (2.6) can be rewritten as

$$\frac{1}{\mathrm{Pr}}\frac{\partial^2\theta(x,t)}{\partial x^2}+g(x,t) =\frac{\partial^\gamma\theta(x,t)}{\partial t^\gamma}, \quad m-1<\gamma\leq m \quad (2.10)$$

Applying Fourier integral sine transform to Eqs. (2.10) and (2.7), we get

$$\frac{\partial^\gamma\theta(\zeta,t)}{\partial t^\gamma}+\frac{1}{\mathrm{Pr}}\zeta^2\theta(\zeta,t) =\sqrt{\frac{2}{\pi}}\frac{1}{\mathrm{Pr}}\zeta+g(\zeta,t) \quad (2.11)$$

$$\theta^{(n)}(\zeta,0)=\sqrt{\frac{2}{\pi}}\int_0^\infty \sin(\zeta x)a_n(x)dx \quad (2.12)$$
$$=a_n(\zeta),\ n\geq 0$$

Using initial condition (2.12) for getting fractional Laplace transform of Eq. (2.11) as

$$p^\gamma\tilde{\theta}(\zeta,p)+\frac{1}{\Pr}\zeta^2\tilde{\theta}(\zeta,t)-\sum_{n=0}^{m}p^{\gamma-n-1}\tilde{\theta}^{(n)}(\zeta,0) \quad (2.13)$$
$$=\sqrt{\frac{2}{\pi}}\frac{1}{\Pr}\frac{\zeta}{p}+\tilde{g}(\zeta,p)$$

Then

$$\tilde{\theta}(\zeta,t)=\sqrt{\frac{2}{\pi}}\frac{1}{\Pr}\frac{\zeta}{p\left(p^\gamma+\frac{\zeta^2}{\Pr}\right)} \quad (2.14)$$
$$+\frac{\tilde{g}(\zeta,p)}{p^\gamma+\frac{\zeta^2}{\Pr}}+\sum_{n=0}^{m}\frac{a_n(\zeta)p^{\gamma-n-1}}{p^\gamma+\frac{\zeta^2}{\Pr}}$$

Taking the inverse Laplace transform of Eq. (2.14) and using the relation

$$L^{-1}\left\{\frac{n!\,p^{\lambda-\mu}}{\left(p^\lambda\mp c\right)^{n+1}}\right\}=t^{\lambda n+\mu-1}E_{\lambda,\mu}^{(n)}\left(\pm ct^\lambda\right),$$
$$\left(\operatorname{Re}(p)>|c|^{\frac{1}{\lambda}}\right)$$

Then Eq. (2.14) leads to

$$\theta(x,t)=\frac{2}{\pi}\int_0^\infty\frac{\zeta}{\Pr}\sin(\zeta x)t^\gamma E_{\gamma,\gamma-1}\left(\frac{-\zeta^2}{\Pr}t^\gamma\right)d\zeta$$

$$+\sqrt{\frac{2}{\pi}}\int_0^\infty\sin(\zeta x)g(\zeta,t)t^{\gamma-1}E_{\gamma,\gamma}\left(\frac{-\zeta^2}{\Pr}t^\gamma\right)d\zeta$$

$$+\sum_{n=0}^{m}\sqrt{\frac{2}{\pi}}\int_0^\infty\sin(\zeta x)a_n(\zeta)E_{\gamma,n+1}\left(\frac{-\zeta^2}{\Pr}t^\gamma\right)d\zeta$$
(2.15)

Special cases:

1. When γ=1, $a_0(\zeta)=0$, then Eq. (2.15) yields

$$\theta(x,t)=\frac{2}{\pi}\int_0^\infty\frac{\zeta}{\Pr}\sin(\zeta x)\exp\left(\frac{-\zeta^2}{\Pr}t\right)$$

$$\times\int_0^t\left[\sqrt{\frac{2}{\pi}}\frac{1}{\Pr}\zeta+g(\zeta,\tau)\exp\left(\frac{-\zeta^2}{\Pr}\tau\right)\right]d\tau\,d\zeta$$
(2.16)

which is the result obtained by Fang and others [2].

2. When γ=1, g(ζ,t)=0, a0(ζ)=0, then from Eq. (2.15) we get

$$\theta(x,t)=1-erf\left(\frac{x}{2\sqrt{\frac{t}{\Pr}}}\right)$$
(2.17)

which is the result obtained also by Fetacau and Corina [3].

3. When $0<\gamma\leq 1$, then Eq. (2.15) yields

$$\theta(x,t)=\frac{2}{\pi}\int_0^\infty\frac{\zeta}{\Pr}\sin(\zeta x)t^\gamma E_{\gamma,\gamma-1}\left(\frac{-\zeta^2}{\Pr}t^\gamma\right)d\zeta$$

$$+\sqrt{\frac{2}{\pi}}\int_0^\infty\sin(\zeta x)g(\zeta,t)t^{\gamma-1}E_{\gamma,\gamma}\left(\frac{-\zeta^2}{\Pr}t^\gamma\right)d\zeta$$

$$+\sqrt{\frac{2}{\pi}}\int_0^\infty \sin(\zeta x)a_0(\zeta)E_{\gamma,1}\left(\frac{-\zeta^2}{\Pr}t^\gamma\right)d\zeta \quad (2.18)$$

which is the result obtained by Salim and El-Kahlout [12].

3. The time-fractional energy equation in xz plane:

The time-fractional energy equation in xz plane is written as

$$\frac{k}{c\rho}\left[\frac{\partial^2\theta(x,z,t)}{\partial x^2}+\frac{\partial^2\theta(x,z,t)}{\partial z^2}\right] + \frac{\nu}{c}f(x,z,t)+\frac{r(x,z,t)}{\rho c} = \frac{\partial^\gamma\theta(x,z,t)}{\partial t^\gamma} \quad (3.1)$$

where

$$f(x,z,t)=\left[\frac{\partial u(x,z,t)}{\partial x}\right]^2+\left[\frac{\partial u(x,z,t)}{\partial z}\right]^2$$ is

a known function as soon as the velocity field u(x,z,t) is prescribed, r(x,z,t) is the radiant heating, which is neglected.

The corresponding initial and boundary conditions of Eq. (3.1) are

$$\frac{\partial^{(n)}\theta(x,z,0)}{\partial t^{(n)}}=a_n(x,z), \quad for \quad x>0\,,\ z>0 \ and\ n\ge 0 \quad (3.2)$$

$$\theta(0,z,t)=\theta(x,0,t)=T_0, \quad for \quad t>0 \quad (3.3)$$

Moreover, the natural condition

$$\theta(x,z,t),\ \frac{\partial^{(n)}\theta(x,z,t)}{\partial x^{(n)}},\ \frac{\partial\theta^{(n)}(x,z,t)}{\partial z^{(n)}}\to 0, \quad (3.4)$$
$$for \quad x^2+z^2\to\infty \ and\ n\ge 0$$

also have to be satisfied.

Using the non-dimensional quantities (2.5), and $z^*=\frac{zU}{\nu}$, Eqs. (3.1), (3.2), (3.3) and (3.4) reduce to dimensionless equations as follows (for brevity the dimensionless mark "*" are omitted here).

$$\frac{1}{\Pr}\left[\frac{\partial^2\theta(x,z,t)}{\partial x^2}+\frac{\partial^2\theta(x,z,t)}{\partial z^2}\right]+ \lambda\left\{\left[\frac{\partial u(x,z,t)}{\partial x}\right]^2+\left[\frac{\partial u(x,z,t)}{\partial z}\right]^2\right\}=\frac{\partial^\gamma\theta(x,z,t)}{\partial t^\gamma} \quad (3.5)$$

where m-1<γ≤m

$$\theta^{(n)}(x,z,0)=a_n(x,z), \quad for \quad x>0\,,\ z>0\ and\ n\ge 0 \quad (3.6)$$

$$\theta(0,z,t)=\theta(x,0,t)=1, \quad for \quad t>0 \quad (3.7)$$

$$\theta(x,z,t),\ \frac{\partial^{(n)}\theta(x,z,t)}{\partial x^{(n)}},\ \frac{\partial^{(n)}\theta(x,z,t)}{\partial z^{(n)}}\to 0, \quad for \quad x^2+z^2\to\infty \quad (3.8)$$

Letting

$$g(x,z,t)=\lambda\left\{\left[\frac{\partial u(x,z,t)}{\partial x}\right]^2+\left[\frac{\partial u(x,z,t)}{\partial z}\right]^2\right\}$$

, then Eq. (3.5) becomes

$$\frac{1}{\Pr}\left[\frac{\partial^2\theta(x,z,t)}{\partial x^2}+\frac{\partial^2\theta(x,z,t)}{\partial z^2}\right]+g(x,z,t)=\frac{\partial^\gamma\theta(x,z,t)}{\partial t^\gamma} \qquad (3.9)$$

By following the same steps as in section 2, we get the exact solution of Eq. (3.9) as

$$\theta(x,z,t)=\frac{4}{\pi^2}\int_0^\infty\int_0^\infty\frac{\zeta^2+\xi^2}{\Pr}\sin(\zeta x)\times\sin(\xi z)t^\gamma E_{\gamma,\gamma-1}\left(-\frac{\zeta^2+\xi^2}{\Pr}t^\gamma\right)d\zeta\,d\xi$$

$$+\frac{2}{\pi}\int_0^\infty\int_0^\infty\sin(\zeta x)\sin(\xi z)g(\zeta,\xi,t)t^{\gamma-1}E_{\gamma,\gamma}\left(-\frac{\zeta^2+\xi^2}{\Pr}t^\gamma\right)d\zeta\,d\xi$$

$$+\frac{2}{\pi}\int_0^\infty\int_0^\infty\sum_{n=0}^{m}\sin(\zeta x)\sin(\xi z)a_n(\zeta,\xi)E_{\gamma,n+1}\left(-\frac{\zeta^2+\xi^2}{\Pr}t^\gamma\right)d\zeta\,d\xi \qquad (3.10)$$

Special case:

When $0<\gamma\le 1$, then Eq. (3.10) yields

$$\theta(x,z,t)=\frac{4}{\pi^2}\int_0^\infty\int_0^\infty\frac{\zeta^2+\xi^2}{\Pr}\sin(\zeta x)\sin(\xi z)t^\gamma\times E_{\gamma,\gamma-1}\left(-\frac{\zeta^2+\xi^2}{\Pr}t^\gamma\right)d\zeta\,d\xi$$
$$+\frac{2}{\pi}\int_0^\infty\int_0^\infty\sin(\zeta x)\sin(\xi z)g(\zeta,\xi,t)t^{\gamma-1}\times E_{\gamma,\gamma}\left(-\frac{\zeta^2+\xi^2}{\Pr}t^\gamma\right)d\zeta\,d\xi$$
$$+\frac{2}{\pi}\int_0^\infty\int_0^\infty\sin(\zeta x)\sin(\xi z)a_0(\zeta,\xi)\times E_{\gamma,1}\left(-\frac{\zeta^2+\xi^2}{\Pr}t^\gamma\right)d\zeta\,d\xi \qquad (3.11)$$

which is the result obtained by Salim and El-Kahlout [12].

3. Conclusion:

This paper has presented some results about the time-fractional energy equation in x and xz plane. Exact solution of this equation is obtained by using the Fourier sine integral transform and integral Laplace transform. The Caputo fractional derivative is considered in time-fractional energy equation as time derivative, where the order of the fractional derivative is considered as $m-1<\gamma\le m$. Special cases have been considered in the cases $\gamma=1,\ 0<\gamma<1$.

REFERENCE [1] Erde`lyi A., Magnus W., Oberhettinger F. and Tricomi F. G., Tables of Integral Transforms. Vol. I , McGraw-Hill, New York, 1954. | [2] Fang S. , Wenchang T., Yaohua Z. and Takashi M. , The Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivative model, J. Math. Phys. 7(2006) 1072-1080. | [3] Fetacau C. and Corina F., The Rayliegh-Stokes problem for a heated second grade fluids. Int. J. Non-Linear Mech. 37(2002) 1011-1015. | [4] Gorenflo R. and Mainardi F.. Fractional calculus: Integral and differential equation of fractional order, in: Carpinteri A, Minardi F. (Eds), Fractals and fractional Calculus in continuum Mechanics, Springer Verlag, Wien and New York, 1997, 223-276. Available from http://www.fracalmo.org. | [5] Langlands T. A. M., Solution of a modified fractional diffusion equation. Physica A, 367 (2006), 136-144. | [6] Liu F., Anh V. V., Turner I. and Zhuang P., The fractional advection-dispersion equation. J. Appl. Math. Computing 13(2003), 233-245. | [7] Mainardi F. and Pagnini G., The Wright functions as solution of the time | fractional diffusion equation. Appl. Math. Comput. 141 (2003), 51-62. | [8] Mainardi.F and Pagini G., The role of the Fox-Wright functions in fractional sub-diffusion of distributed order. J. Comput Appl. Math. 207 (2007) 245-257. | [9] Mainardi.F, Pagini G. and Gorenflo R., Some aspects of fractional diffusion | equations of single and distributed order. Appl. Math. Comput. 187(2007) 295-305. | [10] Podlubny I., Fractional differential equations. Academic Press, New York, 1999. | [11] Saichev A. and Zaslavsky G., Fractional kinetic equations: solutions and applications. Chaos 7(1997), 753-764. | [12] Salim T. and El-Kahlout A., Solution of fractional order Rayleigh stokes equations, Adv. Theor. Appl. Mech., Vol. 1, 2008, no. 5, 241-254. | [13] Saxena R. K., Mathai A. M. and Haubold H. J., Solutions of certain fractional kinetic equations and a fractional diffusion equation.ArXiv:0704.1916v1[math.CA] 15 Apr 2007. | [14] Schneider W. R. and Wyss W., Fractional diffusion equation and wave equations. J. Math. Phys. 30 (1989), 134-144. | [15] Wyss W., The fractional diffusion equation. J. Math. Phys. 27(1986), 2782-2785. | [16] Yu R. and Zhang H., New function of Mittag-Leffler type and its application in the fractional diffusion-wave equation. Chaos, Solutions Fract. (2005), doi: 10. 1016/j. chaos. 2005.08.151. |