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Selection Principles and D-Spaces

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ABSTRACT *The D -property is a covering property. Some special cases where unions of spaces or subspaces that are D make a whole space a D-space is discussed in this paper. The implications between selection principles and D -spaces are also discussed.*

INTRODUCTION

The concept of D-spaces was first introduced in the year 1979 by E.K.Van Douwen and W.F.Pfeffer [1]. The implication between the D-property and the three selection principles namely Rothberger, Menger and Hurewicz is discussed in this paper.

Definition 1.1.[1] A neighbourhood assignment for a topological space (X, τ) is a function $N: X \to \tau$ such that $x \in N(x)$ for each $x \in X$. X is said to be a D-space if for every neighbourhood assignment N, there is a closed discrete subset D of X such that $N(x)\backslash x \in D$ covers X.

Theorem 1.2. If X is the countable union of closed D-subspaces then X is a D-space.

Proof : Let $X = \bigcup_{n<\omega} F_n$ where each F_n is a closed D-subspace of X. Let $\{U(x)/x \in X\}$ be the range space of neighbourhood assignment U. Pick a closed discrete subset D_1 of F_1 such that $\mathfrak{D}_1 = \{U(x)/x \in D_1\}$ covers F_1.Since $\{U(x)/x \in F_2\backslash \bigcup\mathfrak{D}_1$ covers $F_2\backslash \bigcup\mathfrak{D}_1$ which is subset of F_2, pick a closed discrete D_2 of $F_2\backslash \bigcup\mathfrak{D}_1$ such that $\mathfrak{D}_2 = \{U(x)/x \in \mathfrak{D}_2\}$ covers $F_2\backslash \bigcup\mathfrak{D}_1$. Note that $\widetilde{\mathbb{D}_2} = D_1 \bigcup D_2$ is a closed discrete subset of $F_1 \bigcup F_2$ such that $\widetilde{\mathbb{D}_2} = D_1 \bigcup D_2$ covers $F_1 \bigcup F_2$. Inductively since $\{U(x)/x \in F_n(\bigcup\widetilde{\mathfrak{D}_{n-1}})\}$ covers $F_n(\bigcup\widetilde{\mathfrak{D}_{n-1}})$ which is a closed subset of F_n , pick a closed discrete subset D_n of $F_n(\bigcup\widetilde{\mathfrak{D}_{n-1}})$ such that $D_n = \{U(x)/x \in D_n\}$ covers $F_n(\bigcup\widetilde{\mathfrak{D}_{n-1}})$. Note that $\widetilde{\mathbb{D}_n} = \widetilde{\mathbb{D}_{n-1}} \bigcup D_n$ is a closed discrete subset of $F_1 \bigcup F_2 \bigcup \ldots \bigcup F_n$ such that $\widetilde{\mathbb{D}_n} = \widetilde{\mathbb{D}_{n-1}} \bigcup D_n$ covers $F_1 \bigcup F_2 \bigcup \ldots \bigcup F_n$. Letting $D = \bigcup_{n<\omega} D_n$ and $\mathfrak{D} = \bigcup_{n<\omega} \mathfrak{D}_n$ it is seen that D is a closed discrete subset of X such that $\{U(x)/x \in D\}$ covers X.

Corollory 1.3. F_σ subsets of D-spaces are D-spaces.

Proposition 1.4. If $X = Y \bigcup Z$, where Y and Z are D-spaces and Y is closed in X, then X is also a D-space.

Theorem 1.5. If a regular space X is union of a countable family γ of dense metrizable subspaces , then X is a D-space.

Proof : Each $Y \in \gamma$ has a σ disjoints base $\mathfrak{B}_\gamma$. For each $V \in \mathfrak{B}_\gamma$ We fix an open subset $U(V)$ of X such that $U(V) \bigcap Y = V$. For any disjoint elements $V_1$and V_2 of $\mathfrak{B}_\gamma$ the sets $U(V_1)$ and $U(V_2)$ are disjoint , since Y is dense in X . Therefore the family $\mathfrak{B}_\gamma = \{U(V)/V \in \mathfrak{B}_\gamma\}$ is σ - disjoint. Since X is regular, the family $\mathfrak{B}_\gamma$contains a base of X at y, for every $y \in Y$. Hence the family $\mathfrak{P} = \bigcup\{\mathfrak{P}_\gamma/Y \in \gamma\}$ is a σ disjoint base of X. Hence by theorem X is a D-space .

Lemma 1.6. suppose $X = \bigcup\{X_i/i = 1,2, \ldots n\}$ for some $n < \omega$, and let $Y_i =$

$\overline{X_1} \cap \overline{X}_i \cap (X_1 \cup X_i)$, for each $i = 2,3, \dots n$. Then the set $Z = \cup\{Y_i / i = 2,3, \dots n\}$ is closed in .

Proof : Take any $y \in \bar{Z}y \in \bar{Z}$. Then $y \in \bar{y}_i$, for some i, where $2 \le i \le n$

Which implies that $y \in \overline{X_1}$ and $\in \overline{X}_i$. Also $y \in X_k$, for some k, where $1 \le k \le n$. Now we have two cases.

Case 1. $k = 1$. Then $y \in Y_i = \overline{X_1} \cap \overline{X}_i \cap (X_1 \cup X_i) \subset Z, 2 \le i \le n$.

Case 2. $2 \le k \le n$. Then $y \in \overline{X_1} \cap \overline{X_k} \cap (X_1 \cup X_k) = Y_k \subset Z$

Hence, $y \in Z$ and Z is closed in X.

Lemma 1.7. If $X = Y \cup Z$ where each of the subspaces Y and Z has a σ- disjoint base and X is regular, then the subspaces $\overline{Y} \cap \bar{Z}$ also has a σ - disjoint base.

Theorem 1.8. suppose $X = \cup\{X_i / i = 1,2, \dots n\}$ for some $n < \omega$, where X is regular and X_i has a σ - disjoint base, for each $i = 1,2, \dots n$. Then X is a D-space.

Proof: We prove by induction. For $n = 1$ the statement is true, since every space with a point countable space is a D-space. Assume now that for less than n summands the assertion holds. For any i and j such that $1 \le i \le n$ and $1 \le j \le n$ and $i \ne j$ put $Y_{i,j} = \overline{X}_i \cap \overline{X}_j \cap (X_i \cup X_j)$. By Lemma, the set $Z_j = \cup\{Y_{i,j} / i \ne j, 1 \le i \le n\}$ is closed in X. By Lemma each $Y_{i,j}$ is a space with σ- disjoint base. Therefore, the space Z_j is the union of less than n spaces with a σ-disjoint base. By inductive assumption, Z_j is a D-space, for each $j = 1,2, .. n$. Therefore, since each Z_j is a closed in X, the subspace $Z = \cup\{Z_j / j = 1,2, .. n\}$ of X is a D-space.

The family $\mu = \{V_i / 1 \le i \le n\}$, where $V_i = X_i \setminus Z$, is a disjoint family of open subsets of X. Indeed, $X \setminus Z$ is open in X, and no point of x of V_i can belong to the closure of V_j for $i \ne j$, since otherwise x would belong to $Y_{i,j}$ which is contained in Z. Therefore, $X \setminus Z$ has a σ- disjoint base and is a D-space. Hence X is a D-space, as the union of an open D-space and a closed D-space.

Corollory 1.9. If a regular space X is the union of a finite family of metrizable subspaces, then X is a D-space.

Definition 1.10.[2] A subset P of a space X is said to be locally closed if P is open in its closure. i.e, P can be represented as the intersection of an open subset of X with a closed subset of X. For example every discrete subspace of a space X is locally closed in X.

Lemma 1.11. If a space Y is the union of a finite collection of locally closed D-space s, then Y is a D-space.

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Proof: We argue by induction. Let $Y = P_1 \cup P_2 \cup \ldots U P_{n+1}$ where each P_i is locally closed in X and a D-space. Assume that the lemma holds whenever the number of summands does not exceed n. For $i = 1,2..n+1$ put $F_i = \overline{P_i} \backslash P_i$ then F_i is closed in Y and $P_i \cap F_i = \emptyset$. Therefore, F_i is the union of $\leq n$ locally closed D-spaces $P_j \cap F_i$ $i \neq j$. By the inductive assumption it follows that each F_i is a D-space. Clearly, $\overline{P_i} = P_i \cup F_i$. Since F_i is closed in $\overline{P_i}$, and both F_i and P_i are D-spaces, it follows that $\overline{P_i}$ is a D-space, for each $i = 1,2,\ldots n+1$. Hence Y is also a D-space, as the union of a finite number of locally closed D-spaces.

Theorem 1.12. Suppose that $= X_1 \cup \ldots U X_k$, where X is a regular space and each X_i is a space with a point countable base. Then X is a D-space.

Proof : We prove by induction. Assume that the statement is not true for not more than $K - 1$ summands. Put $H_{i,j} = \overline{\overline{X_i} \cap X_j}$,

$W_{i,j} = \overline{X_i} \setminus H_{i,j}$, for $i,j = 1,2,\ldots k$ and $F = \cap\{H_{i,j} / i,j = 1,2,\ldots k\}$. Let $i \neq j$, then obviously, $W_{i,j} \cap X_j = \emptyset$. Therefore $W_{i,j}$ is the union of $\leq k - 1$ spaces with a point countable base and the inductive assumption implies that $W_{i,j}$ is a D-space. $W_{i,j}$ is locally closed in X. The space $V_i = X \backslash \overline{X_i} = X \setminus \overline{X_i}$ is also locally closed in X, for each $i = 1,2,\ldots k$. Since the sets V_i and X_i are disjoint, the space V_i is the union of $\leq k-1$ spaces with a point countable base, and the inductive assumption implies that V_i is a D-space. It follows from the above lemma that the subspace $E = (\cup\{W_{i,j} / i, j = 1,\ldots k\}) \cup (\cup\{V_i / i = 1,..k\})$ of X is a D-space. Take any $x \in X \setminus E$. Then $x \in \overline{X_i}$ and

$x \notin W_{i,j}$ for all $i,j = 1,\ldots,k$. It follows that $x \in H_{i,j}$ for all $i,j = 1,\ldots,k$.

Hence $x \in F$. Thus $X = E \cup F$.

Let us show that the space F has a point countable base. Fix a point countable base B_i in each space $X_i, i = 1,\ldots,k$. for each $V \in B_i$. Let $\Phi(V)$ be the largest open subset of $\overline{X_i}$ such that $\Phi(V) \cap X_i = V$. put $G_i = \{\Phi(V) / V \in B_i\}, G = U\{G_i / i = 1,\ldots,k\}$, and $S = \{W \cap F / W \in G\}$. From regularity of X it easily follows that S is a base of the space F.

Claim : The family S is point countable.

Proof : To prove the claim, we have to show that for any $j = 1,\ldots,k$ and any $x \in F$, the family G_j is countable at x; that is, only countably many elements of G_j contain x. Obviously $x \in X_i$ for some $i = 1,\ldots,k$. Then $x \in F \subset H_{i,j} = \overline{\overline{X_i} \cap X_j}$. However the space $\overline{X_i}$ is first

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countable at the point x, since $x \in \overline{X_i}$ and the space X is regular. It follows that the tightness of the space $\overline{X_i}$ at the point x is countable, and there exists a countable subset $M \subset \overline{X_i} \cap X_j$ such that $\in \overline{M}$. Let γ_x be the family of all elements of G_j containing x. Clearly $x \in \overline{X_j}$, and each $W \in \gamma_z$ is an open neighbourhood of x in $\overline{X_j}$. However M is countable and the family G_j is point countable at each point of M, since $M \subset X_j$ and B_j is a point countable base of the space X_j. Therefore γ_x is countable. It follows that S is a point countable base of F. Hence F is a D-space. Since F is closed in X and $X = E \cup F$ where E is also a D-space, X is a D-space.

Corollory 1.13. If a regular space X of countable extent is the union of a finite collection of subspaces with a point countable base , then X is Lindelof.

Definition 1.14.[3] A space X is said to be locally D , if every point $x \in X$ has a neighbourhood U such that U with the relative topology of X is a D-space.

Theorem 1.15. If a locally D-space λ is the union of a finite collection of neighbourhood spaces then λ is a D-space.

Theorem 1.16. If a locally D-space λ is the union of a finite collection of neighbourhood and locally compact then λ is a D-space.

Definition 1.17.[8] An Alster cover G of a topological space X is a cover by G_δ sets such that every compact subset of X is included in a member of G. A space is said to be Alster if every Alster cover has a countable subcover.

Definition 1.18.[4] A space X is said to be Menger if for each sequence $\{U_n\}_{n<w}$ of open covers, such that each finite union of elements of U_n is a member of U_n, there are $u_n \in U_n$, $n < w$, such that $\{ u_n \,/\, n < w\}$ is an open cover.

Definition 1.19.[5] A space X is said to be Rothberger if for each sequence $\{U_n\}_{n<w}$ of open covers, there are $u_n \in U_n$ such that $\{ u_n \,/\, n < w\}$ is an open cover.

From the definition it can be seen that every Rothberger space is Menger.

Definition 1.20.[6] A γ over of a space X is a countably infinite open cover such that each point $x \in X$ is in all but finitely many members of the cover. A space X is said to be Hurewicz if given a sequence $\{ u_n \,/\, n < w\}$ of γ-covers, there is for each , a finite $V_n \subseteq U_n$ such that either $\{ \cup V_n \,/\, n < w\}$ is a γ-cover, or else for some n, V_n is a cover.

Lemma 1.21. Every Menger space is D.

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Theorem 1.22. Every Hurewicz space is Menger.

Proof : Let $\{U_n\}_{n<w}$ be a sequence of open covers, such that each finite union of elements of U_n is a member of U_n. Rewriting the sequence of open covers $\{U_n\}_{n<w}$ as $\{U_{i<n}U_i\}_{n<w}$ we get a sequence of γ-covers. Since X is Hurewicz there is for each n, a finite $V_n \subseteq U_n$ such that either $\{ \cup V_n \ / \ n < w\}$ is a γ-cover, or else for some n, V_n is a cover. Hence there are $V_n \in U_n$ such that $\{ V_n \ / \ n < w\}$ is an open cover.

Corollary 1.23. Every Hurewicz space is D.

Theorem 1.24. Every Alster space is Menger.

Proof : Let $\{U_n\}_{n<w}$ be a sequence of open covers of X, each closed under finite unions. Let G be the set of all $\cap_{n<w}, U_n's$, where $u_n \in U_n$. Let K is included in some $u_n \in U_n$. Thus K is included in some $G' \in G$. Since X is Alster , there are $\{H_k\}_{k<w}$ in G such that $\cup_{k<w} H_k$ covers X. Let $H_k = \cap_{n<w}, U_{nk}$, where $U_{nk} \in U_n$. Then $\{U_{nn}\}_{n<w}$ covers , since $H_n \subseteq U_{nn}$. Thus since each $U_{nn} \in U_n$, X is Menger.

Corollory 1.25. Every Alster space is D.

Theorem 1.26. If X is concentrated on a Rothberger subspace, then X is Rothberger.

Definition 1.27.[8] A space is said to be Lusin if every nowhere dense set is countable.

Corollory 1.28. Every seperable Lusin space is Rothberger and therefore D.

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