Construction of Three Associate Class PBIB Designs using Method of Juxtapositions

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ABSTRACT *Two Series of three associate class Partially Balanced Incomplete Block (PBIB) Designs using a method of juxtaposing some association matrices are constructed in this paper. The designs used as base designs are four associate class PBIB designs with fixed replications, but in new series there is no restriction on the parameters of PBIB designs. New series are constructed by juxtaposing some association matrices of a series of four associate class PBIB designs of partial geometry (r,k,t) with finite graphs. Association schemes of both the series, P- matrices of the association scheme, efficiencies factors along with overall efficiency factor and illustrations of both the series have also been given in this paper.*

1. Introduction

In the construction of PBIB designs with higher associate classes, our emphasis will always be that the number of associate classes needed to be reduced so that we can estimate various treatment affects with more accuracies. Keeping this mind the existing four associate class PBIB designs have been reduced to new three associate class PBIB designs by using the method of juxtaposing some association matrices. In such a way, the new patterned matrix raises three associate class PBIB designs with higher efficiencies.

Clatworthy [7] lists complete description on two associate class PBIB designs. Sinha.et.al [6], Kageyama.et.al [3,4,5] introduced some patterned constructions of PBIB designs. Recently, Garg,et.al [2] have also introduced some patterned constructions of PBIB designs by using juxtaposing some association matrices along with their association schemes.

Here, we have introduced two series of three associate class PBIB designs by reducing an already existing series of four associate class PBIB designs used by partial geometry (r,k,t) with finite graphs introduced by Garg,et.al[1]. Association scheme of series-I is based on occurrence of treatment pairs and association scheme of series-II is based on occurrence of binary results (present/absent) of a treatments combination under consideration in the incidence matrix.

2. Base Series for constructing new series of three associate class PBIB designs

Let us first, consider an already existing series of four associate class PBIB designs constructed by using partial geometry (r,k,t) with finite graphs with parameters

$v = s^2$, $b = 3s$, $r = 3$, $k = s$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\lambda_4 = 0$, $n_1 = n_2 = n_3 = s-1$ and $n_4 = (s-1)(s-2)$ ------(1)

follows an association scheme given by Garg,et.al[1]. PBIB designs with parameters stated above are used for constructing two new series of PBIB designs with three associate classes.

3. Main Theorems
Theorem 3.1.

Let B_0, B_1, B_2, B_3 and B_4 denotes the association matrices a series of four associate class PBIB designs of partial geometry (r,k,t) with finite graphs having parameters defined in equation (1). If we consider the patterned matrix

$N^* = [B_{1d} \mid B_{2d}]'$

Here $B_{1d} = B_0 + B_1$ and $B_{2d} = B_0 + B_2$

Then N^* denotes the incidence matrix of a series of three associate class PBIB designs having parameters for ($s \geq 3$)

$v = 2s^2$, $b = s^2$, $r = s$, $k = 2s$, $\lambda_1 = s$, $\lambda_2 = 1$, $\lambda_3 = 0$, $n_1 = s-1$, $n_2 = s^2$, $n_3 = s(s-1)$ -------(2)

follows an association scheme - 3.1 given below.

Proof:
Since N^* denotes the incidence matrix of a series of three associate class PBIB designs, so it contains zeros and ones only. The order of N^* matrix is $2s^2 \times s^2$, where ($s \geq 3$)

The product $N^*N^{*\prime}$ can be written as

$$N^*N^{*\prime} = [\,B_{1d} \mid B_{2d}\,]' [\,B_{1d} \mid B_{2d}\,] \quad \text{or}$$

$$N^*N^{*\prime} = \begin{bmatrix} B_{1d}^2 & B_{1d}B_{2d} \\ B_{2d}B_{1d} & B_{2d}^2 \end{bmatrix}$$

Clearly, $N^*N^{*\prime}$ provides the following observations

1) Each row sum in $N^*N^{*\prime}$ is same.
2) Each column sum in $N^*N^{*\prime}$ is same.
3) Each row sum is also equal to each column sum
4) Each diagonal position of $N^*N^{*\prime}$ is also unique and equal to 's'.

Noting these observations, we say that $N^*N^{*\prime}$ is a treatment structure matrix of a PBIB design. Since each row sum of patterned matrix N^* is 's' taken as replication of each treatment as well as each column sum of patterned matrix N^* is '2s' taken as block size. Also, every diagonal element of $N^*N^{*\prime}$ is equal to 's' shows a replication size of a treatment of PBIB design.

Hence, N^* denotes the incidence matrix of a series of three associate class PBIB designs with parameters $v = 2s^2$, $b = s^2$, $r = s$, $k = 2s$, $\lambda_1 = s$, $\lambda_2 = 1$, $\lambda_3 = 0$, $n_1 = s-1$, $n_2 = s^2$, $n_3 = s(s-1)$ following association scheme defined below.

3.1.1. Association Scheme – 3.1
Association scheme is based on occurrence of treatment pairs. Let us we take a particular treatment say 'θ'. Treatment pairs of type $(\theta\theta_i)$ occurs 's' times in these blocks, where i = 1,2,…,s-1 are 1st associate of 'θ' and $n_1 = s-1$. Treatment

pairs of type $(\theta \; \theta_j)$ occurs only once in these blocks , where j= 1,2,…,s^2 are 2nd associate of 'θ' and $n_2 = s^2$.

Remaining treatments which do not occur with 'θ' in any block or in other words, treatments pair of type $(\theta \theta_k)$ does not occur in these blocks. All these φ_k's where k = 1,2,…,s(s-1) are 3rd associates of θ and n_3 =s(s-1).

3.1.2. P- Matrices

$$P_1 = \begin{bmatrix} s-2 & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & s(s-1) \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0 & s-1 & 0 \\ s-1 & 0 & s(s-1) \\ 0 & s(s-1) & 0 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 0 & 0 & s-1 \\ 0 & s^2 & 0 \\ s-1 & 0 & s(s-2) \end{bmatrix}$$

3.1.3. Illustration

In equation (2), if s=3, we have a three associate class PBIB design with parameters

v= 18 , b= 9, r=3 , k= 6, $\lambda_1 = 3$, $\lambda_2 = 1$, $\lambda_3 = 0$, $n_1 = 2$, $n_2 = 9$, $n_3 = 6$. From equation (1), we get the base design with parameters v= 9, b=12, r=3, k= 3, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\lambda_4 = 0$, $n_1 = n_2 = n_3 = 2$ and $n_4 = 2$. The method of construction of blocks, association scheme along with P- matrices are discussed as below.

Here, let B_0 , B_1 , B_2 , B_3 and B_4 denotes the association matrices of four associate PBIB design. Consider a patterned matrix

$$N^* = \begin{bmatrix} B_{1d} \\ B_{2d} \end{bmatrix}$$

Here $B_{1d} = B_0 + B_1$ and $B_{2d} = B_0 + B_2$

The order of N* matrix is18 x 9 and N* N*' satisfies all the properties of a matrix to be treatment structure matrix. Each row sum of N* is '3' and each column sum is '6' as well as every diagonal element is '3', which shows that replication of each treatment is '3' and each block consist of exactly '6' treatments. Collection of numbers against ones from column-wise gives treatments of a particular block and collection of numbers against ones from row – wise gives replication of a particular treatment.

In this way, N* denotes the incidence matrix of a series of three associate class PBIB design with parameters

v= 18 , b= 9, r=3 , k= 6, $\lambda_1 = 3$, $\lambda_2 = 1$, $\lambda_3 = 0$, $n_1 = 2$, $n_2 = 9$, $n_3 = 6$

and the blocks of this design are

(1,2,3,10,13,16)   (1,2,3,11,14,17)   (1,2,3,12,15,18)

(4,5,6,10,13,16)   (4,5,6,11,14,17)   (4,5,6,12,15,18)

(7,8,9,10,13,16)   (7,8,9,11,14,17)   (7,8,9,12,15,18)

Now, we explain the association scheme – 3.1 along with P - matrices.

Let us we take a particular treatment say 'θ=1'. Treatment pairs of type $(\theta \; \theta_j)$ which are (1,2) and (1,3) occurring three times in these blocks are 1st associate of '1' and $n_1 = 2$.

Treatment pairs of type $(\theta \; \theta_j)$, where θ_j's are 10,11,12,13,14,15,16,17 and 18 occurs only once in these blocks are 2nd associate of '1' and $n_2 = 9$.

Treatment pairs type $(\theta \; \theta_k)$, where θ_k's are 4,5,6,7,8,9 does not occur with '1' in any block are 3rd associates of '1' and n_3 =6.

P- Matrices are

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 6 \\ 0 & 6 & 0 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 9 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

Theorem 3.2.

Let us consider the association matrices B_0 , B_1 , B_2 , B_3 and B_4 of a four associate class PBIB designs constructed by using partial geometry (r,k,t) with finite graphs with parameters defined in equation (1). Also, consider the patterned matrix

N* = B3c - B4 =Bs2

Then patterned matrix N* denotes the incidence matrix of a series of three associate class PBIB designs with parameters

v= s^2 = b, r =2s-1 = k, λ_1 = s, λ_2 = 2 = λ_3, n_1 = 2(s-1), n_2 = s-1 , n_3 = (s-1)(s-2).   -------- (3)

and B_3c is obtained from B_3 by interchanging zero's and one's for (s≥3).

Proof:

Since matrix N* denotes the incidence matrix of a series of three associate class PBIB

designs so it will contains zeros and ones only. Here, N* is a square matrix of order s^2. Clearly, the product matrix N*N*' shows that each row sum and column sum of N*N*' is unique and same for every row and each column. The diagonal elements of N*N*' are also unique and represent replications of a PBIB design. In this way, N*N*' represent a treatment structure matrix of a new design in which rows correspond to treatment and columns correspond to blocks. Row sum and column sum of matrix N* is 2s-1, so each treatment is replicated '2s-1' times and each block contain '2s-1' treatments. Hence, patterned matrix N* denotes the incidence matrix of a series of three associate class PBIB designs with parameters

v= s^2 = b, r =2s-1 = k, λ_1 = s, λ_2 = 2 = λ_3, n_1 = 2(s-1), n_2 = s-1 , n_3 = (s-1)(s-2)

follows an association scheme – 3.2 defined below.

3.2.1. Association scheme – 3.2

Each row/ column contains zeros and ones only. This association is based on result of binary number (one/zero) in row – wise. Let us consider a treatment 'θ' (say) in a particular row.

Treatments having '1's in the same row are first associates to each other and n_1 = 2(s-1). Treatments in a row under consideration showing '0 's due to the structure of matrix B_3C in patterned matrix Bs^2 are second associates of 'θ' and n_2=s -1.

Remaining treatments in a row under consideration showing '0 's due to the structure of matrix B_4 in Bs^2 are third associates of 'θ' and n_3=(s -1)(s-2).

3.2.2. P- Matrices

$$P_1 = \begin{bmatrix} s-2 & 1 & s-2 \\ 1 & 0 & s-2 \\ s-2 & s-2 & (s-2)(s-3) \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 2 & 0 & 2(s-2) \\ 0 & s-2 & 0 \\ 2(s-2) & 0 & (s-2)(s-3) \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 2 & 2 & 2(s-3) \\ 2 & 0 & s-3 \\ 2(s-3) & s-3 & s^2-6s+10 \end{bmatrix}$$

3.2.3. Illustration

In equation (3), if s=3, we have a three associate class PBIB designs with parameters.

$v= 9 = b$, $r = 5 = k$, $\lambda_1 = 3$, $\lambda_2 = 2 = \lambda_3$, $n_1 = 4$, $n_2 = 2$, $n_3 = 2$. The base design can be generated from equation (1) with parameters

$v= 9$, $b=12$, $r=3$, $k= 3$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\lambda_4 = 0$, $n_1 = n_2 = n_3 = 2$ and $n_4 = 2$.

The method of construction blocks, association scheme along with P- matrices are discussed as below.

Let B_0 , B_1 , B_2 , B_3 and B_4 denotes the association matrices of four associate PBIB design. Consider the patterned matrix

$$N^* = B_3 c - B_4 = B_9$$

N^* is a square matrix of order 9. Clearly, in $N^*N^{*\prime}$ each row sum and each column sum is unique and same and each diagonal element of $N^*N^{*\prime}$ is '5' which shows that $N^*N^{*\prime}$ shows treatment structure matrix of a new design in which rows corresponding to treatment and columns corresponding to blocks. Row sum and column sum of N^* is '5', so each treatment is replicated '5' times and each block contain '5' treatments. In this way, N^* denotes the incidence matrix of a series of three associate class PBIB designs having parameters $v= 9= b$, $r = 5 = k$, $\lambda_1 = 3$, $\lambda_2 = 2 = \lambda_3$, $n_1 = 4$, $n_2 = 2$, $n_3 = 2$ and blocks of this design are

(1,2,3,4,7)	(1,2,3,5,8)	(1,2,3,6,9)
(1,4,5,6,7)	(2,4,5,6,8)	(3,4,5,6,9)
(1,4,7,8,9)	(2,5,7,8,9)	(3,6,7,8,9)

Association scheme – 3.2 is based on result of binary number (one/zero) in row – wise.

Let us consider a treatment '$\theta=1$' (say) in a first row of incidence matrix. Treatments 2,3,4,7 having '1' 's in first row. It means 2,3,4,7 are present in first row, so these are 1st associates of '1' and $n_1 = 4$. Treatments 5 and 9 in this row shows '0' 's means absent in this row due to the structure matrix $B_3 c$ in the patterned matrix B_9 are 2nd associates of '1' and $n_2=2$. Treatments 6 and 8 in this also shows '0' 's in first row due to the structure matrix B_4 in the patterned matrix B_9 are 3rd associates of '1' and $n_3 = 2$. The P – matrices of this design are

$$P_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Table - 1. Three Associate Class PBIB Designs with r ≤ 9, k ≤ 18 from theorem 3.1

s	v	b	r	k	λ_1	λ_2	λ_2	E_1	E_2	E_2	E
3	18	9	3	6	3	1	0	1	.818	.75	.809
4	32	16	4	8	4	1	0	1	.842	.80	.837
5	50	25	5	10	5	1	0	1	.862	.833	.859
6	72	36	6	12	6	1	0	1	.878	.857	.876
7	98	49	7	14	7	1	0	1	.890	.875	.889
8	128	64	8	16	8	1	0	1	.901	.888	.900
9	162	81	9	18	9	1	0	1	.910	.90	.909

Table - 2. Three Associate Class PBIB Designs with r , k ≤ 15 from theorem 3.2

s	v	b	r	k	λ_1	λ_2	λ_2	E_1	E_2	E_2	E
3	9	9	5	5	3	2	2	.916	.876	.876	.886
4	16	16	7	7	4	2	2	.932	.890	.890	.907
5	25	25	9	9	5	2	2	.944	.904	.904	.917
6	36	36	11	11	6	2	2	.952	.915	.915	.925
7	49	49	13	13	7	2	2	.958	.924	.924	.932
8	64	64	15	15	8	2	2	.962	.932	.932	.938

Conclusion

Here, we have constructed two series of three associate class PBIB designs by reducing already existing four associate class of PBIB designs. The base design has fixed replication (r = 3), but in newly constructed series, there is no restriction on any parameter of PBIB design. Seven PBIB designs with r ≤ 9 , v ≤ 162 are listed in table 1 and six PBIB designs for r ≤ 15, v ≤ 64 listed in table-2 . In both series, number of treatments increase very rapidly due to square power of 's'. In Table-I, efficiencies of treatments which are first associates to each other is exactly 100% even for less number of replications. Table-II represents symmetric PBIB designs having higher replications. Therefore, efficiencies of newly constructed three associate class PBIB designs are much better than four associate class PBIB designs of base series and quite useful for comparing the efficiencies with existing three associate class PBIB designs. We observe that new series of designs are adjustable for particular parametric combinations.

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