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Results of Non Homogeneous Indefinite Quadratic Forms

KEYWORDS

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Let $Q(X_1, X_2, \ldots, X_n)$ be a real indefinite quadratic form in n variables of determinant $D \neq 0$ and of type (r, n-r), $0<r<n$. Let $s = 2r - n$ denote the signature of Q. Blaney [1], has shown that there exists a constant $\Gamma>0$, independent of Q and depending only on n and r, such that given any real numbers $C_1,\ldots,C_n$ there exist integers $X_1,\ldots, X_n$ satisfying

$$0< Q (X_1 + C_1, \ldots,X_n + C_n) \leq (\Gamma |D|)^{1/n}. \qquad (1)$$

Let $\Gamma_{r,n-r}$ denote the infimum of all such number Γ.

The forms for which equality is necessary in (1) with Γ replaced by forms for which equality is necessary in (1) with Γ replaced by $\Gamma_{r,n-r}$ and for some $C_1,\ldots, C_n$ are called critical forms. The problem of finding positive inhomogeneous minimum is to determine $\Gamma_{r,n-r}$ for various n and r and to find the critical forms.

In this notation the following results are known

$\Gamma_{1,1} = 4$, Davenport-Heilbronn [2],

$\Gamma_{2,1} = 4$, Blaney[3] , and Barnes [4],

$\Gamma_{1,2} = 8$, $\Gamma_{3.1} = 16/3$, $\Gamma_{2.2 = 16}$, Dumir [5], [6], [7],

$\Gamma_{1.3} = 16$, Dumir and Hans-Grill [8],

$\Gamma_{3.2} = 16$, $\Gamma_{4.1} = 8$, Hans- Gill and Madhu Raka [9], [10],

$\Gamma_{r.n-r} = 2^n/(s+1)$ where $s = 2r-n = 0,1,2,3$, Bambah, Dumir and Hans-Gill [12],[14].

$\Gamma_{2,3} = (7/4)^5$, $\Gamma_{r,r+1} = 2^{n-1}$, for $r \geq 3$, Bambah, Dumir and Hans-Gill Han [11]

Bambah, Dumir and Hans-Gill conjectured that (see [12] and [13]),

$$\Gamma_{r,n-r} = \begin{cases} \dfrac{2^n}{|s_0|+1} & if\ s \equiv s_0 (mod 8)\ and\ |s_0| \leq 3 \\ \dfrac{2^n}{4} & if\ s \equiv 4 (mod 8) \end{cases}$$

with possibly some exceptions.

Flahive [15] proved the conjecture of Bambah, Dumir and Hans-Gill for $n \geq 21$.

Aggarwal and Gupta [16],[17],[18], determined $\Gamma_{r,r+2}$ and $\Gamma_{r,r+3}$ for r 3, and $\Gamma_{r+4,r}$ for $r \geq 1$, proving there by the conjecture of Bambah, Dumir and Hans-Gill in these cases.

Ranjeet Sehmi and Dumir [19] proved that $\Gamma_{2,5} = 32$.

$\Gamma_{2,4}$ was obtained by Dumir, Hans Gill and Sehmi [20].

It follows from the work of Dumir, Hans-Gill and Woods[13], that for $n \geq 6$, $\Gamma_{r,n-r}$ depends only on n and signature s (mod8). Thus all values of $\Gamma_{r,n-r}$ are known except for $\Gamma_{1,4}$. It may be remarked here that $\Gamma_{r,n-r}$ is easier to obtain when the number of variables is large. It is conjectured that $\Gamma_{1,4} = 8$. From Jackson [21] , Margulis [22] and Watson[23], it follows that $\Gamma_{1,4}$ 32. In 1994 Dumir and Sehmi [24], showed that $8 \leq \Gamma_{1,4} \leq 16$. Madhu Raka and Urmila Rani [25] improve the upper bound to get that $8 \leq \Gamma_{1,4} < 12$.

Another problem of study is of isolation of Γ Davenport and Heilbronn [2], proved that $\Gamma_{1,1}$ is not isolated i.e. if Q is not equivalent to the critical forms at which $\Gamma_{1,1}$ is attained and of $0 < \Gamma < \Gamma_{1,1,}$ then the inequality

$$0 < Q (X_1 + C_1, X_2 + C_2) \leq (\Gamma |D|)^{1/2}. \qquad (2)$$

Is not solvable in integers x_1, x_2 for some Q and for some real c_1, c_2. A form is called incommensurable if it is not a multiple of a rational form. It follows from Margulis [22] that an incommensurable form in n 3 variables, takes arbitrarily small values. Watson [23] had proved that if a form in n 3 variables takes arbitrarily small values, then the inequality (1) is solvable with arbitrarily small Γ. For rational forms and $n \geq 3$ Vulakh [26] had proved that $\Gamma_{r,n-r}$ is isolated i.e. if Q is not equivalent to the critical forms, then (1) is solvable for some Γ, $0 < \Gamma < \Gamma_{r,n-r.}$

If $\Gamma^{(k)}_{r,n-r}$ denote the k[th] successive inhomogeneous minima for positive values of indefinite quadratic forms of type (r,n-r), n 3, then = $\Gamma^{(1)}_{r,n-r} = \Gamma^{\square}_{r,n-r}$.

In this notation Bambah, Dumir and Hans-Gill [11] proved that $\Gamma^{(2)}_{2,3} = 16$. Dumir and Hans-Gill [27] obtained $\Gamma^{(2)}_{1,2} = 27/4$. Flahive [15] has obtained the second minima for forms of signature $0,\pm1,\pm2$ in $n \geq 21$ variables. In 1922 Dumir and Sehmi [28],[29] obtained $\Gamma^{(2)}_{3,2} = 8$ and extended this result to obtain $\Gamma^{(2)}_{r+1,r} = 2^{n-2}$ for all $r \geq 2$. Raka Madhu [30], obtained the first four minima for ternary forms of type (2,1) for the class of zero forms. Madhu Raka and Urmila Rani [31], [32] proved that $\Gamma^{(2)}_{2,1} = 8/3$, and $\Gamma^{(2)}_{3,1} = 4$. They [33] also proved that $\Gamma^{(2)}_{r+2,r} = 2^{2r}$ for all $r \geq 1$. Madhu Raka and Urmila Rani [34] obtained first five minima for positive values of zero quaternary quadratic forms of type (2,2)

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