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Common Coupled Fixed Point Theorems in Metric Spaces With W-Distance

KEYWORDS: Metric space, Common coupled fixed points, family of mappings, w-distance .

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ABSTRACT *In this paper, we use the concept of a w-distance to prove a common coupled fixed point theorem for a family of maps on a metric space. Our results unify, generalize and complement the comparable results from the current literature.*

1. INTRODUCTION

Presence of fixed points is an intrinsic property of a self-map. A point which remains invariant under a transformation is called a fixed point. The presence or absence of a fixed point is an intrinsic property of a map. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach's contraction principle which gives an answer on the existence and uniqueness of a solution of an operator equation Tx = x; is the most widely used fixed point theorem in all of analysis. This principal is constructive in nature and is one of the most useful tools in the study of nonlinear equations. In 1996; Kada et al. [9] introduced the notion of a w-distance. They elaborated, with the help of examples, that the concept of a w-distance is more general than that of metric on a nonempty set. They also proved a generalization of the Caristi fixed point theorem employing the definition of a w-distance on a complete metric space. Recently, Ilic and Rakocevic [8] obtained fixed point and common fixed point theorems in terms of a w-distance on a complete metric spaces. The notion of coupled fixed points was introduced by Chang and Ma [4]. Since then, the concept has been of great interest to many researchers in metric fixed point theory. Motivated by recent works on coupled fixed points [1, 2] as well as some fixed point theorems of Ciric [3], we obtain a common coupled fixed point theorem for a family of maps on metric spaces. Our results generalize and extend some recently announced ones in the literature.

In order to do this, we recall some definitions and lemmas.

Definition 1.1.

Let (X, d) be a metric space. An element $(x, y) \in X \times X$ is said to be a coupled fixed point for the mappings $T: X \times X \to X$ if $T(x, y) = x$ and $T(y, x) = y$.

Definition 1.2.

Let (X, d) be a metric space with metric d. A function $p: X \times X \to [0, +\infty)$ is called a w-distance on X if

(1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;

(2) p is lower semi- continuous in its second variable. i.e. if $x \in X$ and $y_n \to y$ in X then $p(x, y) \leq \liminf_n p(x, y_n)$.

Lemma 1.1. [9]. Let (X, d) be a metric space and p be a w- distance on X. If $\{x_n\}$ is a sequence in X such that $\lim_n p(x_n, x). = \lim_n p(x_n, y)$, then x = y. In particular $p(z, x) = p(z, y) = 0$ then $x = y$.

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Lemma 1.2. [9]. Let p be a w- distance on a metric space (X, d) and let $\{x_n\}$ be a sequence in X such that $\in > 0$ there exists $N_\in \in N$ such that $m > n > N_\in$ implies $p(x_{n,} x_m) < \in$ (or $\lim_{m,n\to\infty} p(x_{n,} x_m) = 0$), then $\{x_n\}$ is a Cauchy sequence.

2. MAIN RESULT

Theorem 2.1. Let p be a w- distance on a complete metric space (X, d) and let $\{T_\alpha\}_{\alpha\in J}$ with $T_\alpha: X \times X \to X$ be a family of mappings. If there exists $\beta \in J$ such that for each $\alpha \in J$,

$$p\left(T_\alpha(x,y), T_\beta(u,v)\right) \le a \max\left\{p\left(x, T_\alpha(x,y)\right), p\left(u, T_\beta(u,v)\right)\right\}, \quad (1)$$

and

$$p\left(T_\alpha(x,y), T_\beta(u,v)\right) \le a \max\left\{p\left(x, T_\beta(x,y)\right), p\left(u, T_\alpha(u,v)\right)\right\}, \quad (2)$$

for all $x, y, u, v \in X$, where $a < 1$, then the family $\{T_\alpha\}_{\alpha\in J}$ has a unique common coupled fixed point.

Proof. Choose $(x_0, y_0) \in X \times X$ and in general, consider $x_{2n+1} = T_\alpha(x_{2n}, y_{2n})$, $y_{2n+1} = T_\beta(y_{2n}, x_{2n})$, $x_{2n+2} = T_\beta(x_{2n+1}, y_{2n+1})$, $y_{2n+2} = T_\alpha(y_{2n+1}, x_{2n+1})$. Then, we have by (1),

$$p(x_{2n+1}, x_{2n+2}) = p\left(T_\alpha(x_{2n}, y_{2n}), T_\beta(x_{2n+1}, y_{2n+1})\right),$$

$$\le a \max\left\{p\left(x_{2n}, T_\alpha(x_{2n}, y_{2n})\right), p\left(x_{2n+1}, T_\beta(x_{2n+1}, y_{2n+1})\right)\right\}$$

$$= a \max\{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\}$$

and so $p(x_{2n+1}, x_{2n+2}) \le ap(x_{2n}, x_{2n+1})$. (3)

Next, from (2), we have

$$p(x_{2n}, x_{2n+1}) = p\left(T_\beta(x_{2n-1}, y_{2n-1}), T_\alpha(x_{2n}, y_{2n})\right),$$

$$\le a \max\left\{p\left(x_{2n-1}, T_\beta(x_{2n-1}, y_{2n-1})\right), p\left(x_{2n}, T_\alpha(x_{2n}, y_{2n})\right)\right\}$$

$$= a \max\{p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1})\}$$

and so $p(x_{2n}, x_{2n+1}) \le ap(x_{2n-1}, x_{2n})$. (4)

It now follows from (3) and (4), that

$$p(x_n, x_{n+1}) \le ap(x_{n-1}, x_n) \le a^n p(x_0, x_1)$$

for n = 1,2,…….Hence,

$$p(x_n, x_{n+r}) \le p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \ldots p(x_{n+r-1}, x_{n+r})$$

$$\le (a^n + a^{n+1} + \cdots a^{n+r-1}) p(x_0, x_1)$$

$$\le \frac{a^n}{1-a} p(x_0, x_1),$$

for $n, r = 1, 2, \ldots \ldots$ Since $a < 1$, it follows that $\{x_n\}$ is a Cauchy sequences in X and since (X, d) is a complete metric space, there exist $x^* \in X$ such that $\lim_{n\to\infty} x_n =$

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x^*. It follows similarly that $\{y_n\}$ is a Cauchy sequences in X with a limit $y^* \in X$.

We now show that (x^*, y^*) is common coupled fixed point of T_α for each $\alpha \in J$. Using (2) we have,

$$p(x^*, T_\alpha(x^*, y^*))$$
$$\leq p(x^*, x_{2n})$$
$$+ p(T_\beta(x_{2n-1}, y_{2n-1}), T_\alpha(x^*, y^*))$$
$$\leq p(x^*, x_{2n}) + a \max \{p(x_{2n-1}, T_\beta(x_{2n-1,} y_{2n-1})), p(x^*, T_\alpha(x^*, y^*))\}$$
$$= p(x^*, x_{2n}) + a \max\{p(x_{2n-1}, x_{2n}), p(x^*, T_\alpha(x^*, y^*))\}.$$

Letting $n \to \infty$, we get,

$$p(x^*, T_\alpha(x^*, y^*)) \leq ap(x^*, T_\alpha(x^*, y^*)).$$

Hence $p(x^*, T_\alpha(x^*, y^*)) = 0$ and so $T_\alpha(x^*, y^*) = x^*$. Next, we have

$$p(y^*, T_\alpha(y^*, x^*))$$
$$\leq p(y^*, y_{2n+1})$$
$$+ p\left(T_\beta(y_{2n}, x_{2n}), T_\alpha(y^*, x^*)\right)$$
$$\leq p(y^*, y_{2n+1}) + a \max \{p(y_{2n}, T_\beta(y_{2n}, x_{2n})), p(y^*, T_\alpha(y^*, x^*))\}.$$

Letting $n \to \infty$, we get,

$$p(y^*, T_\alpha(y^*, x^*)) \leq ap(y^*, T_\alpha(y^*, x^*))$$

and so $T_\alpha(y^*, x^*) = y^*$. Hence (x^*, y^*) is a coupled fixed point of T_α.

We now have to prove that (x^*, y^*) is unique common coupled fixed point of T_α for each $\alpha \in J$. Suppose that (x^1, y^1) is a second coupled fixed point of T_α. Then by (1), we obtain

$$p(x^1, x^*) = p\left(T_\alpha(x^1, y^1), T_\beta(x^*, y^*)\right),$$
$$\leq a \max\{p(x^1, T_\alpha(x^1, y^1)), p\left(x^*, T_\beta(x^*, y^*)\right) = 0,$$

From which it follows that $x^1 = x^*$. It follows similarly that $y^1 = y^*$. Hence (x^*, y^*) is unique common coupled fixed point of T_α for each $\alpha \in J$.

Corollary 2.1. Let p be a w- distance on a complete metric space (X, d) and let $\{T_\alpha\}_{\alpha \in J}$ with $T_\alpha: X \times X \to X$ be a family of mappings. If there exists $\beta \in J$ such that for each $\alpha \in J$,

$$p\left(T_\alpha(x, y), T_\beta(u, v)\right)$$
$$\leq a \max\{p(x, u), p(y, v)\},$$

for all $x, y, u, v \in X$, where $a < 1$, then the family $\{T_\alpha\}_{\alpha \in J}$ has a unique common coupled fixed point

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