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Equivariant Estimation of Uniform Location Model

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ABSTRACT *LEHMANNAND CASELLA (1998) provide a detailed discussion on equivariant estimation of the parameters of location, scale, location-scale models. EDWIN PRABAKARAN and CHANDRASEKAR (1994) developed simultaneous equivariant estimation approach and illustrated the method with examples urtherLEO ALEXANDER(2000) studied the simultaneous Equivariant estimation of the parameters in invariant models based on various censored samples. In this paper, we consider uniform models and obtain minimum risk equivariant estimators of the parameters based on type II censored samples.*

1. Introduction

Equivariance is a desirable property used for restricting the class of estimators whenever the model possesses symmetry. ZACKS (1971) and LEHMANNAND CASELLA (1998) provide a detailed study of the problem of equivariant estimation for certain models. In the case of location-scale model, LEHMANNAND CASELLA (1998) develops marginal Equivariant procedure for estimating the parameters.EDWIN PRABAKARAN and CHANDRASEKAR (1994) have proposed a simultaneous Equivariant estimation for estimating the parameters of a location-scale model. For a detailed discussion on simultaneous equivariant estimation and related results the reader is referred to EDWIN PRABAKARAN (1995).

In this paper, by invoking the above procedures, we obtain optimal estimators for the parameter(s) of uniform model under Type II censoring. The paper is organized as follows: section 2 deals with the problem of Equivariant estimation for the uniform location model considering three different loss functions namely Squared error loss function, Absolute error loss function and Linex loss function.

1.1 Preliminaries

Suppose N randomly selected units were placed on a test simultaneously, the failure times of the first n units to fail were observed. Thus the number of completely determined life spans is n and the number of censored ones is (N-n). let$X_{i:N}$, i=1,2,…,n denote the failure times of the completely observed items. Then the joint probability density function (pdf) of $(X_{1:N}, X_{2:N},..., X_{n:N})$ (BAIN, 1978) is

$$g_\theta(x_1, x_2,..., x_n) = \frac{N!}{(N-n)!}\prod_{i=1}^{n} f_\theta(x_i)\left\{1 - F_\theta(x_i)\right\}^{N-n} \quad (1.1)$$

Here f_θ and F_θ denote the common pdf and the distribution function of the failure times of the units selected randomly, which are put to test. Further n is assumed to be known in advance.

2. Location model

Suppose $\mathbf{X} = (X_1, X_2,..., X_N)'$ is a random sample from uniform population with joint pdf

$$f(x-\xi) = \begin{cases} 1\ ,\ \xi \le x_i \le \xi + 1\ , i = 1,2,.......N;\ \ \xi \in R \\ 0, otherwise. \end{cases}$$

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Let $X_{1:N} \le X_{2:N} \le ... \le X_{n:N}$ be the times of the first n units to fail and assume that the experiment was stopped as soon as the n-th item failed. The joint pdf of the first n observations, according to (1.1), is given by

$$g_{\xi}(x_{1:N},...,x_{n:N}) = \frac{N!}{(N-n)!}\{1-(x_{n:N}-\xi)\}^{N-n}$$

$$, \ \xi \le x_{1:N} \le x_{n:N} \le \xi+1; \xi \in R. \quad ...(2.1)$$

Note that the above pdf belongs to a location model. We are interested in deriving MRE estimator of ξ by considering three loss functions.

Case (i) : Suppose the loss is squared error, we obtain MRE estimator of ξ. Take $\delta_0(\mathbf{X}) = (X_{1:N} + X_{n:N})/2.$ Cleary δ_0 is an equivariant estimator which is a function of the sufficient statistics $(X_{1:N}, X_{n:N})'$ Further δ_0 is not complete sufficient. Since we are interested in the evaluation of conditional distribution under $\xi = 0$, take $\xi = 0$ in (2.1). In order to find $v^* = E_0(\delta_0 | \mathbf{y})$, consider the transformation

$$Y_1 = (X_{1:N} + X_{n:N})/2 \text{ and}$$
$$Y_i = X_{i:N} - X_{1:N} \quad , i = 2,3,...,n.$$

Now

$$X_{1:N} = Y_1 - Y_n/2 \quad and$$
$$X_{i:N} = Y_i + Y_1 - Y_n/2 \quad , i = 2,3,...,n.$$

The Jacobian of the transformations is J=1. Thus the joint pdf of $Y = (Y_1, Y_2, ..., Y_n)'$ is given by

$$h(y_1,...,y_n) = \frac{N!}{(N-n)!}(1-y_1-y_n/2)^{N-n} \ ,$$
$$0 < y_2 < y_3 < ... < y_n,$$
$$y_n/2 < y_1 < 1-y_n/2, \ 0 < y_n < 1.$$

Also, the joint pdf of $(Y_2,...,Y_n)$ is given by

$$h_1(y_2,...,y_n) = \frac{N!}{(N-n)!}\int_{y_n/2}^{1-y_n/2}(1-y_1-y_n/2)^{N-n}\,dy_1$$
$$= \frac{N!}{(N-n+1)!}(1-y_n)^{N-n+1},$$
$$0 < y_2 < y_3 < ... < y_n < 1.$$

Thus the conditional pdf of $\delta_0 = Y_1 \ given \ (Y_2,...,Y_n)$ is given by

$$h^*(y_1 | y_2,...,y_n) = (N-n+1)\frac{(1-y_1-y_n/2)^{N-n}}{(1-y_n)^{N-n+1}} ,$$
$$y_n/2 < y_1 < 1-(y_n/2). \quad ...(2.2)$$

Now

$$v^* = E_0(\delta_0 | \mathbf{y}) \qquad (2.3)$$
$$= \frac{(N-n+1)}{(1-y_n)^{N-n+1}}\int_{y_n/2}^{1-y_n/2} y_1(1-y_1-y_n/2)^{N-n}\,dy_1$$

Setting $y_1 - (y_2/2) = (1-y_n)u$, we get $y_1 = (y_n/2) + (1-y_n)u$,

So,

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$$v^* = \frac{(N-n+1)}{(1-y_n)^{N-n+1}}\int_0^1 \left[\begin{array}{l}\{y_n/2+(1-y_n)u\}\times \\ (1-y_n)^{N-n+1}(1-u)^{N-n}\end{array}\right] du$$
$$= \frac{y_n}{2} + \frac{(1-y_n)}{(N-n+2)} .$$

Therefore the MRE estimator of ξ is given by

$$\delta^*(\mathbf{X}) = \xi_0(X) - E_0(\delta_0 \mid \mathbf{y})$$
$$= \frac{X_{1:N}+X_{n:N}}{2} - \frac{X_{n:N}-X_{1:N}}{2} - \frac{1-X_{n:N}+X_{1:N}}{2}$$
$$= \frac{(N-n+1)X_{1:N}+X_{n:N}-1}{(N-n+2)} \quad ...(2.4)$$

Moreover, when the loss is squared error, the MRE estimator $\delta^*(\mathbf{X})$ can be evaluated more explicitly by the Pitman form (LEHMANN AND CASELLA 1998 p.154).

Therefore the Pitman estimation of ξ, in view of (2.1), is given by

$$\delta^*(\mathbf{X}) = \frac{\int_{X_{n:N}-1}^{X_{1:N}} u\, f(X_{1:N}-u,...,X_{n:N}-u)\,du}{\int_{X_{n:N}-1}^{X_{1:N}} f(X_{1:N}-u,...,X_{n:N}-u)\,du} .$$

$$= \frac{\int_{X_{n:N}-1}^{X_{1:N}} u\,(1-X_{n:N}+u)^{N-n}\,du}{\int_{X_{n:N}-1}^{X_{1:N}} (1-X_{n:N}+u)^{N-n}\,du}$$
$$= \frac{(N-n+1)X_{1:N}+X_{n:N}-1}{(N-n+2)} .$$

This estimator coincides with the one given in (2.4).

Remark 2.1. If n=N, the above estimator reduces to

$$\delta^*(\mathbf{X}) = (X_{1:N}+X_{n:N}-1)/2 ,$$

which is same as the estimator obtained for the complete sample case (LEHMAN AND CASELLA, 1998).

Case (ii): If the loss is absolute error, then we obtain MRE estimator of ξ by considering δ_0 = median of conditional distributions of $\delta_0(\mathbf{X})$ given $\mathbf{Y}=\mathbf{y}$. Take $\delta_0(\mathbf{X}) = (X_{1:N}+X_{n:N})/2.$,so that $v_0 = 1/2$.
Therefore the MRE estimator of ξ is given by

$$\delta^*(\mathbf{X}) = (X_{1:N}+X_{n:N})/2 - 1/2$$
$$= \frac{X_{1:N}+X_{n:N}-1}{2} .$$

Case (iii): Consider the location invariant Linex loss function (VARIAN, 1975) .

$$L(\xi:\delta) = e^{a(\delta-\xi)} - a(\delta-\xi) - 1 , \quad a \in R-\{0\} .$$

In order to find v^*, take $\delta^*(\mathbf{X}) = (X_{1:N}+X_{n:N})/2$, consider

$$R(\delta \mid \mathbf{y}) = E_0[\{e^{a(\delta_0-v)} - a(\delta_0-v)-1\} \mid \mathbf{y}]$$
$$= e^{-av}E_0(e^{a\delta_0} \mid \mathbf{y}) - aE_0(\delta_0 \mid \mathbf{y}) + av - 1$$
$$= e^{-av}E_0(e^{a\delta_0} \mid \mathbf{y}) - a\left\{\frac{X_{n:N}-X_{1:N}}{2} + \frac{1-X_{n:N}+X_{1:N}}{(N-n+2)}\right\} + av - 1,$$

in view of (2.3).

Define $M(a;\mathbf{y}) = E_0(e^{a\delta_0} \mid \mathbf{y})$.

$$\textit{Then} \quad \frac{dR}{dv} = e^{-av}(-a)M(a;\mathbf{y}) + a$$

$$\textit{and} \quad \frac{d^2R}{dv^2} = e^{-av}a^2 M(a;\mathbf{y})$$

$$\textit{Note that} \quad \frac{d^2R}{dv^2} > 0 \quad \forall v$$

$$\textit{Thus} \quad \frac{dR}{dv} = 0 \Rightarrow \quad e^{-av} M(a;\mathbf{y}) = 1.$$

$$\textit{So that } v^* = \frac{\log M(a;\mathbf{y})}{a} = \frac{1}{a}\{cgf \ of \ (\delta_0 \mid \mathbf{y}) \ at \ a\}.$$

Therefore the MRE estimator of ξ is givenby

$$\delta^*(\mathbf{X}) = \frac{X_{1:N} + X_{n:N}}{2} - \frac{1}{a}\{cgf \ of \ (\delta_0 \mid \mathbf{y}) \ at \ a\}.$$

If n=N, then from (2.2)

$$v^* = \frac{1}{a}\{\log(e^a - 1) - \frac{ay_N}{2} - \log a - \log(1 - y_N)\}$$
$$= \frac{1}{a}\left\{\log\left(\frac{(e^a - 1)}{a(1 - y_N)}\right) - \frac{ay_N}{2}\right\}.$$

Therefore the MRE estimator of ξ is given by

$$\delta^*(\mathbf{X}) = \frac{X_{1:N} + X_{n:N}}{2} - \frac{1}{a}\left\{\log\left(\frac{(e^a - 1)}{a(1 - y_N)}\right) - \frac{ay_N}{2}\right\}.$$

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