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On The Classification of Root Systems Up to Their Cartan Matrices

KEYWORDS

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ABSTRACT *The root systems are known to provide a relatively uncomplicated way of completely characterizing simple and semi-simple Lie algebras. The goal of this paper is to show that root systems may be themselves completely characterized 'up to isomorphism' by their Cartan matrices.*

1. Preliminaries:-

i. A Lie algebra may be understood as a vector space with an additional bilinear operation known as the **Commutator** [,] defined for all elements and satisfying certain properties.

ii. A Lie algebra is called **simple** if it's only ideals are itself and 0, and specifically the derived algebra : $\{ [x,y] | x,y \in \mathfrak{g}\} = [\mathfrak{g},\mathfrak{g}] \neq 0$.

iii. Let the Lie algebra $\mathfrak{g}$ be **semi-simple** decomposable as the direct product of simple Lie algebra.

vi. a Lie algebra $\mathfrak{g}$ is called **nilpotent** if there exists a decreasing finite sequence $(\mathfrak{g}_i)_{i\in[0,k]}$ of ideals such that $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_k = 0$ and $[\mathfrak{g},\mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$ for all $i \in [0, k-1]$.

v. Given a real Lie algebra $\mathfrak{g}_R$ the **Killing form** on $\mathfrak{g} \times \mathfrak{g}$ is defined by

$$B(X,Y) = -Tr(ad\, X \circ ad\, Y) \in R$$

2. Cartan sub-algebras:

Definition (2.1):

A Cartan sub-algebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is nilpotent Lie sub-algebra that is equal to its centralizer, such that $\{X \in \mathfrak{g}: [X,\mathfrak{h}] \subset \mathfrak{h}\} = \mathfrak{h}$.

For semi-simple Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ being Cartan is equivalent to $\mathfrak{h}$ being a maximal abelian sub-algebra.

3.Root decomposition and root systems:-

Definition (3.1):

A root system is finite **set** of non-zero vectors $\Delta \subseteq \mathbb{E}$ satisfies the following :

(R1) If $\alpha \in \Delta$, then $\lambda\alpha \in \Delta$ if and only if $\lambda = \pm 1$

(R2) If $\alpha, \beta \in \Delta$, then $\sigma_\alpha . \beta \in \Delta$ where $\sigma_\alpha : \mathbb{E} \to \mathbb{E}$is reflection

Each element of Δ is called a root.

Theorem (3-2):-

1- We have the following decomposition for $\mathfrak{g}$, called the root decomposition

$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha = \{x \mid [h,x] {}^{i}= \langle\alpha,h\rangle x$ for all $h \in \mathfrak{h}$ }

$R = \{ \in \mathfrak{h}^* - \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$

The set is called the root system of $\mathfrak{g}$, and sub spaces $\mathfrak{g}_\alpha$ are called the root sub spaces.

2- $[\mathfrak{g}_\alpha , \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ (here and below, we let $\mathfrak{g}_0 = \mathfrak{h}$)

3- If $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha , \mathfrak{g}_\beta$ are orthogonal with respect to the Killing form K.

4- For any α, the Killing form gives a non-degenerate pairing $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to \mathbb{C}$. in particular, restriction of K to $\mathfrak{h}$is non-degenerate.

Example (3-3):-

Let $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{h}$ = diagonal matrices with trace 0. Denote by $e_i : \mathfrak{h} \to \mathbb{C}$ the functional which computes i^{th} diagonal entry of h:

$$e_i : \begin{bmatrix} h_1 & 0 & \dots . \\ 0 & h_2 & \dots . \\ 0 & \cdots & h_n \end{bmatrix} \mapsto h_i$$

Then one easily sees that $\sum e_i = 0$, so

$\mathfrak{h}^* = \bigoplus \mathbb{C}\, e_i \,/\, \mathbb{C}(e_1 + \cdots\cdots\cdots + e_n)$.

It is easy to see that matrix units E_{ij} are eigen vectors for ad$h$, $h \in \mathfrak{h}$: $[h, E_{ij}] = (h_i - h_j)E_{ij} = (e_i - e_j)(h)E_{ij}$.

Thus, the root decompstion is given by

$R = \{e_i - e_j \mid i \neq j\} \subset \bigoplus \mathbb{C}\, e_i \,/\, \mathbb{C}(e_1 + \cdots\cdots\cdots + e_n)$.

$\mathfrak{g}_{e_i - e_j} = \mathbb{C}E_{ij}$.

The Killing form on $\mathfrak{h}$ is given by

$(h, h') = \sum_{i \neq j} (h_i - h_j)(h'_i - h'_j) = 2n \sum_i h_i h'_i = 2n \text{tr}(hh')$

From this, it is easy to show that if $\lambda = \sum \lambda_i e_i$, $\mu = \sum \mu_i e_i \in \mathfrak{h}^*$, and λ_i , μ_i are chosen so that $\sum \lambda_i = \sum \mu_i = 0$ (which is always possible), then the corresponding form on $\mathfrak{h}^*$ is given by

$(\alpha, \mu) = \frac{1}{2n} \sum_i \lambda_i \mu_i$

Lemma (3-4):-

1.Let$\alpha \in R$, then $(\alpha, \alpha) = (H_\alpha, H_\alpha) \neq 0$.

2. Let $\in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$ be such that $(e, f) = \frac{2}{(\alpha,\alpha)}$,and let $h_\alpha = \frac{2 H_\alpha}{(\alpha,\alpha)}$

Then $\langle h_\alpha, \alpha \rangle = 2$ and the elements e, f, h_α satisfy the relations of Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. We will denote such a sub algebra by $\mathfrak{sl}(2, \mathbb{C})_\alpha \subset \mathfrak{g}$.

Proof:-

Assume that $(\alpha, \alpha) = 0$; then $\langle H_\alpha, \alpha \rangle = 0$. Choose $\in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$ such that $(e, f) \neq 0$ (from definition (3-1)) . Let $h = [e, f] = (e, f)\ H_\alpha$ and consider the algebra $\mathfrak{a}$ generated by $, f, h$.

then we see that $[e, h] = \langle h, \alpha \rangle\, e = 0$,$[h, f]$= -$\langle h, \alpha \rangle f = 0$, so $\mathfrak{a}$ is solvable Lie algebra . from Lie theorem, we can choose a basis in $\mathfrak{g}$ such that operators ad e, ad$f$, ad$h$ are upper triangular. Since $h = [e, f]$,ad$h$ will be strictly upper-tringular and thus nilpotent. But since $h \in \mathfrak{h}$, it is also semisimple. Thus $h = 0$. On the other hand, $h = (e, f) H_\alpha \neq 0$. This contradiction proves the first part of the theorem.

The second part is immediate from definitions and lemma (3.4).

Theorem (3-5):-

Let $\mathfrak{g}$ be a complex semi simple Lie algebra with Cartan sub algebra $\mathfrak{h}$ and root decomposition

$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$.

1\ R spans $\mathfrak{h}^*$ as a vector space, and elements $h_\alpha, \alpha \in R$, span $\mathfrak{h}$ as a vector space

2\ For each $\alpha \in R$, the root sub space $\mathfrak{g}_\alpha$ is one-dimensional.

3\ For any two roots α, β the number $\langle h_\alpha, \alpha \rangle = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is integer .

4\ For $\alpha \in R$, define the reflection operator $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$s_\alpha(\lambda) = \lambda - \langle h_\alpha, \lambda \rangle \alpha = \lambda - \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} \alpha$$

Then for any roots α, β, $s_\alpha(\beta)$ is also a root. In particular, if $\alpha \in R$, then $-\alpha = s_\alpha(\alpha) \in R$.

5\ For any root α , the only multiples of α which are also roots $\pm\alpha$.

6\ For roots $\alpha, \beta \neq \pm\alpha$, the subspace $V = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\beta + k\alpha}$, is an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})_\alpha$.

7\ If α, β are roots such that $\alpha + \beta$ is also a root, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\beta + k\alpha}$.

Proof:-

1\ Assume that R does not generate $\mathfrak{h}^*$; then there exists a non-zero $h \in \mathfrak{h}$ such that $\langle h, \alpha \rangle = 0$ for all $\alpha \in R$. But then root decomposition (1) implies that ad$h$ = 0. However, by definition in a semi simple Lie algebra, the center is trivial: $\mathfrak{z}(\mathfrak{g}) = 0$.

The fact that h_α span $\mathfrak{h}$ now immediately follows: using identification of $\mathfrak{h}$ with $\mathfrak{h}^*$ given by the Killing form, elements h_α are identified with non-zero multiples of α.

2\ Immediate from Lemma (3-4) and the fact that in any irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$, weight sub spaces are one-dimensional.

3\ Consider $\mathfrak{g}$ as a representation of $\mathfrak{sl}(2, \mathbb{C})_\alpha$. Then elements of $\mathfrak{g}_\beta$ have weight equal to $\langle h_\alpha, \alpha \rangle$. But from the fact that (V admits a weight decomposition with integer weights $V = \bigoplus_{n \in \mathbb{Z}} V[n]$) weights of any finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ are integer.

4\ Assume that $\langle h_\alpha, \alpha \rangle = n \geq 0$. Then elements of $\mathfrak{g}_\beta$ have weight n with respect to action of $\mathfrak{sl}(2, \mathbb{C})_\alpha$. By the same fact above , operator f_α^n is an isomorphism of the space of vectors of weight n with the space of vectors of weight $-n$. In particular, it means that if $v \in \mathfrak{g}_\beta$ is non-zero vector, then$f_\alpha^n\, v \in \mathfrak{g}_{\beta - n\alpha}$ is also non-zero. Thus $\beta - n\alpha = s_\alpha(\beta) \in R$.

5\ Assume that α and $\beta = c\alpha$, $c \in \mathbb{C}$ are both roots. By part (3) ,$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = 2c$ is integer, so c is a half-integer . same argument shows that $1/c$ is also a half-integer. It is easy to see that this implies that $c = \pm 1, \pm 2, \pm 1/2$. Interchanging the roots if necessary and possibly replacing α by $-\alpha$, we have $c = 1$ or $c = 2$.

Now let us consider the sub space

$$V = \mathbb{C}\, h_\alpha \oplus \bigoplus_{k \in \mathbb{Z}, k \neq 0} \mathfrak{g}_{k\alpha} \subset \mathfrak{g}$$

From Lemma (3-4) V is an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})_\alpha$, and by part (2) ,

$V[2] = \mathfrak{g}_\alpha = \mathbb{C} e_\alpha$. Thus, the map ad $e_\alpha : \mathfrak{g}_\alpha \to \mathfrak{g}_{2\alpha}$ is zero. But the results of representation of $\mathfrak{sl}(2, \mathbb{C})$ show that in an irreducible representation, kernel of e is exactly the highest weight sub space. Thus, we see that V has highest weight 2: $V[4] = V[6] = \cdots = 0$.

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This means that $V = \mathfrak{g}_{-\alpha} \oplus \mathbb{C}\, h_\alpha \oplus \mathfrak{g}_\alpha$, so the only integer multiples of α which are roots are $\pm\alpha$. In particular, 2α is not a root.
Combining these two results, we see that if $\alpha, c\alpha$ are both roots, then $c = \pm 1$.
6\ Proof is immediate from dim $\mathfrak{g}_{\beta+k\alpha} = 1$.
7\ We already know that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\beta+k\alpha}$.since dim$\mathfrak{g}_{\beta+k\alpha} = 1$, we need to show that for non-zero $e_\alpha \in \mathfrak{g}_\alpha, e_\beta \in \mathfrak{g}_\beta$, we have $[e_\alpha, e_\beta] \neq 0$. This follows from the previous part and the fact that in an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$, if $v \in V[k]$ is non-zero and $V[k+2] \neq 0$, then $e.v \neq 0$.

Definition (3.6):
A root system is irreducible if it cannot be decomposed into the union of two root systems of smaller rank.

Example(3.7):
Let us Classify all systems of rank 2 , observe that

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)}\frac{2(\alpha,\beta)}{(\beta,\beta)} = 4cos^2\theta$$

Where θ is the angle between α and, this must be an integer, thus there are not many choices for ? θ

$\cos\theta$	0	$\pm\frac{1}{2}$	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{\sqrt{3}}{2}$
θ	$\frac{\pi}{2}, \frac{\pi}{3}$	$\frac{2\pi}{3}, \frac{\pi}{4}$	$\frac{3\pi}{4}, \frac{\pi}{6}$	$\frac{\pi}{6}, \frac{5\pi}{6}$

Choose two vectors with minimal angle between them. If the minimum angle is $\frac{\pi}{2}$, the system is reducible. (notice that α and βcan be scaled independently). If the minimal angle is smaller than $\frac{\pi}{2}$, then $r_\beta(\alpha) \neq \alpha$, so the difference$\alpha - r_\beta(\alpha)$ is non-zero integer multiple of β. (in fact, a positive multiple of β since $\theta < \frac{\pi}{2}$).

4.The Weyl group :

Definition (4.1) :
Choose a base Δ for Φ. Then the simple reflections are defined to be $\sigma_{\alpha_i}$where $\alpha_i$are the simple roots (elements of Δ).

Lemma (4.2):
If$\alpha \in \Delta$, the simple reflection σ_α sends α to $-\alpha$, $-\alpha$ to α and permutes all of the other positive roots.
Proof:
Suppose that $\beta$is a positive root not equal to α. Then $\beta$is not equal to a scalar
multiple of α. So, in the expansion of βas a positive linear combination of simple roots:
$\beta = \sum k_i\alpha_i$ where, say, $\alpha = \alpha_1$, one of the other coefficients, say $k_2 > 0$. Then
$\sigma_\alpha(\beta) = \beta - \langle\alpha,\beta\rangle\alpha = (k_1 - \langle\alpha,\beta\rangle)\alpha_1 + k_2\alpha_2 + \cdots \cdots + k_n\alpha_n$is a positive root since $k_2 > 0$.

Definition (4.3):
If we given a root system $\Delta = \{\alpha_1, \ldots\ldots\ldots, \alpha_N\}$, we call the group generated by the r_{α_i}'s the Weyl group , denoted $\mathcal{W}$. which consists all reflections r_α generated by elements α of root system.
For a given root , the reflection r_α fixes the hyperplane normal to α and maps $\alpha \to -\alpha$. and we can write it as $r_\alpha(\beta) = \beta - \langle\alpha,\beta\rangle\alpha$.
The hyperplanes fixes by the elements of $\mathcal{W}$ partition Einto Weyl chambers. for a given base Δ of E, the unique Weyl chamber containing all vectors γ such that :
$(\gamma,\alpha) \geq 0\ \forall\alpha \in \Delta$,is called the fundamental Weyl chamber.

Proposition (4.4):
Let$\mathcal{W}$ be a crystallographic reflection group in a finite dimensional real vector space. Then, there is a root system Φ in V with Weyl group $\mathcal{W}$.

Proof:
Note that if $\mathcal{W}$ is irreducible, then the root system Φ is unique up to
isomorphism if and only if $\mathcal{W}$ is not of type $B_n, n \leq 3$.
Let Cbe a chamber of $\mathcal{W}$ with walls $L_1, \ldots\ldots\ldots, L_n$. Then, there is a unique root $\alpha_i \in \Phi$orthogonal to $L_i$and lying in the same half-space delimited by $L_i$as C.
The set $\Delta = \{\alpha_i\}_{1\leq i\leq n}$is called a basis of Φ.
Let $\Phi^+ = \{\alpha \in \Phi | \alpha = \sum\alpha_i n_i\ , n \geq 0\}$, (the positive roots) and
$\Phi^- = \{\alpha \in \Phi | \alpha = \sum\alpha_i n_i\ , n \leq 0\}$, (the negative roots).

Lemma (4.5) :
Let$C$ be the set of all $x \in E$ with the property that $(x,\beta) > 0$ for all positive roots β. Then C is a Weyl chamber.We call C the fundamental chamber.

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Proof:

Clearly C is convex and therefore connected. Also $C$is disjoint from all hyperplanes $\beta^{\perp}$. Therefore,C is contained in some Weyl chamber C_0. Suppose that $y \in C_0$ then, since C_0 is connected, there is a path $\gamma(t)$ in C_0 connecting $x \in C$ to. This path does not cross any of the hyper planes. Therefore, by the intermediate value theorem, the sign of $(\gamma(t), \beta)$ remains unchanged. Since it starts as positive, it remains positive. So, $\in C$, proving that $C = C_0$ is a Weyl chamber.

5. Dynkin Diagram:

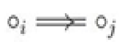
Definition (5.1):

The Dynkin diagram of root system of rank n is defined to be a graph with n vertices labeled with the simple roots α_i and with edges satisfying:

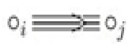
1.No edge connected roots α_i, α_j if they are orthogonal (equivalently, if $c_{ij} = 0$)

2. A single edge connecting α_i, α_j if α_i, α_j are roots of the same length which are not orthogonal (equivalently, $c_{ij} = c_{ji} = -1$)

3. A double edge pointing from α_i to α_j if α_i, α_j are not perpendicular and $\|\alpha_i\|^2 = 2\|\alpha_j\|^2$



4. A triple edge pointing from α_i to α_j if α_i, α_j are not perpendicular and $\|\alpha_i\|^2 = 3\|\alpha_j\|^2$



6. Cartan Matrix:

Definition (6.1):

A Cartan matrix is an $n \times n$ matrix (A_{ij}) with integer coefficients which satisfies the conditions:

i.$A_{ii} = 2$, $i = 1,2, \dots, n$,

ii.$A_{ij} \leqq 0$ if $i \neq j$,

iii.$A_{ij} = 0$ if and only if $A_{ji} = 0$

We say that (A_{ij}) has a null root if there exists a non-zero column vector $[d_i] = [d_1, d_2, \dots, d_n]$ such that $(A_{ij})[d_i] = 0$, where each d_i is non-negative integer. We call (A_{ij}) symmetrizable if there exists a non-singular diagonal matrix Dsuch that the product $(A_{ij})D$ is a symmetric matrix.

We represent Cartan matrices by diagrams which are a slight modification of the diagrams introduced by Coxeter to classify the discrete groups generated by reflections.

Diagrams with weighted arrows (6.2):

We represent the $n \times n$ matrix (A_{ij}) by a diagram in the following way:

(i) The diagram has n vertices.

(ii) For $i \neq j$ we draw $|A_{ij}|$ arrows from the vertex j to vertex i. Each such arrow will be called a (j, i)-arrow.

(iii) To simplify the diagram, when $|A_{ij}| = |A_{ji}| = 1$ we simply draw a line from i to j.

Cartan matrix is called indecomposable if the corresponding diagram is connected.

Indecomposable Cartan matrices with null-roots(6.3) :

A null root is by definition a non-negative solution of the homogeneous system of linear equations :

$\sum_{i=1}^{n} A_{ij}\, x_i = 0\,, \qquad i = 1,2, \dots, n.$

Because $A_{ii} = 2$ and $A_{ij} \leqq 0$ if $[d_1, d_2, \dots, d_n]$ is a null root we have

$\sum_{j \neq i} |A_{ij}|\, d_i = 2d_i \;\; , i = 1,2, \dots, n.$

Finite Cartan matrices (6.4):

Let V_0 be a vector space over the rational field Q with basis $\alpha_0, \alpha_1, \dots, \alpha_n$ and V the subspace spanned by $\alpha_1, \dots, \alpha_n$. Given an $n \times n$ Cartan matrix we define linear transformationsS_i, S_i^* ,$1 \leqq i \leqq n$, acting on V by $\alpha_j S_i = \alpha_j - A_{ij}\alpha_i$ and $\alpha_j S_i^* = \alpha_j - A_{ji}\alpha_i$ introduce a pairing

$$(.,.): V \times V \to Q$$

Defined on our basis by $(\alpha_i, \alpha_j) = A_{ji}$. It is immediate that $\alpha S_i = \alpha - (\alpha, \alpha_i)\alpha_i$ and $\alpha S_i^* = \alpha - (\alpha_i, \alpha)\alpha_i$, and its follow from this that S_i, S_i^* are reflections on V. That is $S_i^2 = id = {S_i^*}^2$ and S_i, S_i^* fix a hyperplane of V pointwise.

Let W (respectivelyW^*) denote the group generated by the elements S_i (respectively S_i^*) for $1 \leqq i \leqq n$. W is called Weyl group of (A_{ij}) so that W^* is Weyl group of the Cartan matrix $(A_{ij})^t$ where t denotes transpose. Notice that $(\alpha S_k, \beta S_k^*) = (\alpha, \beta)$ for $\alpha, \beta \,\varepsilon\, V$ and hence by iteration

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$\left(\alpha S_{i_1},\dots,S_{i_r},\beta S_{i_1}^*,\dots,\beta S_{i_r}^*\right)=(\alpha,\beta)$ for $\alpha,\beta\ \varepsilon\ V, r \geqq 1$ and arbitrary indices $i_1,\dots,i_r\varepsilon\{1,\dots,n\}$.

Definition (6.5):

Let $\left(A_{ij}\right)$ be an $n\times n$ Cartan matrix and W its Weyl group. The elements of the set

$$\Delta=\{\alpha_i\omega|1\leqq i\leqq n,\omega\ \varepsilon\ W\}$$

are called the roots of $\left(A_{ij}\right)$, and Δ is called root system of $\left(A_{ij}\right)$. If the Cartan matrix for which Δ is finite then we call finite Cartan matrix.

Conclusion:

We have shown that there exists a one-to-one correspondence between root systems and Cartan matrices, Given a root system $(\alpha_i), 1\leq i\leq n$, we assign a Cartan matrix by $\left(\alpha_{ij}\right)=\langle\alpha_i,\alpha_j\rangle$.

Conversely, for a Cartan matrix A_{ij}, the root system corresponding to it is assign by Δ described as in definition (6.5).

Moreover we determined the diagram corresponding to a Cartan matrix.

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