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ON THE CONVERGENCE OF THE JUNGCK ITERATION SCHEME FOR APPROXIMATING COMMON FIXED POINTS

KEYWORDS: Jungck iteration, error estimates, point of coincidence, common fixed point, cone metric spaces.

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ABSTRACT *In this paper, we prove a convergence theorem with error estimates for the Jungck iteration process for approximating point of coincidence and common fixed points of three selfmappings of a cone metric space. The result is new even in the classical metric space setting.*

1. INTRODUCTION

In 1976, Jungck [9] obtained a common fixed point theorem for commuting mappings which extends the famous Banach contraction principle. He proved his result by using a new iteration process which is a generalization of the Picard iteration scheme.

Definition 1.1 (Jungck iteration scheme). Let T, S and f be selfmappings of a nonempty set X, and let $(x_n)_{n=0}^{\infty}$ be a sequence in X such that

$$f x_{n+1} = \begin{cases} Tx_n & \text{if } n \text{ is even}, \\ Sx_n & \text{if } n \text{ is odd}, \end{cases} \qquad n = 0,1,2,\ldots$$

Then the sequence $(f x_n)_{n=0}^{\infty}$ is called a (T,S)-f-*sequence* or *Jungck iteration*.

Let f and g be two selfmappings of a nonempty set X. Recall that a point $x \in X$ is called a *coincidence point* of f and g if $fx = gx$. A point $y \in X$ is called a *point of coincidence* of f and g if there exists a point $x \in X$ such that $y = fx = gx$. Mappings f and g are called *weakly compatible* [10] if they commute at their coincidence points.

In 2011, Zhang [17] introduced a useful binary relation between an ordered vector space Y and the set of all subsets of Y.

Definition 1.2 [17]. Let $(Y, \preceq)$ be an ordered vector space, $x \in Y$ and $A \subset Y$. Then we say that $x \preceq A$ if there exists at least one vector $y \in A$ such that $x \preceq y$.

For brevity, throughout the paper we denote by $\mathrm{A}(T,S,f,x,y)$ and $\mathrm{B}(T,S,f,x,y)$ the sets

$$\left\{ d(fx,fy), d(fx,Tx), d(fy,Sy), \frac{d(fx,Sy)+d(fy,Tx)}{2} \right\},$$

$$\left\{ d(fx,fy), \frac{d(fx,Tx)+d(fy,Sy)}{2}, \frac{d(fx,Sy)+d(fy,Tx)}{2} \right\},$$

respectively, where T,S,f are selfmapings of a cone metric space X and $x,y \in X$.

Analogously, $\mathrm{M}(T,S,f,x,y)$ stands for the set

$$\{d(fx,fy), d(fx,Tx), d(fy,Sy), d(fx,Sy), d(fy,Tx)\}.$$

In what follows we denote by I the identity map of a cone metric space X.

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In 2011, Olalelu [12] obtained the following result for common fixed points of three mappings.

Theorem 1.3 (Olalelu [12]). Let (X,d) be a cone metric space over a solid Banach space $(Y,\preceq)$, and T,S and f be selfmappings of X such that $T(X)\cup S(X)\subset f(X)$. Suppose that:

(a) $d(Tx,Ty)\preceq\lambda\,\mathrm{A}(T,S,F,x,y)$ for all points $x,y\in X$ with $x\neq y$, where $\lambda\in[0,1)$.

(b) $d(Tx,Sx)\neq d(fx,Tx)+d(fx,Sx)$ for every $x\in X$ whenever $Tx\neq Sx$.

(c) $f(X)$ or $T(X)\cup S(X)$ is a complete subspace of X.

Then T,S,f have a unique point of coincidence $\xi\in X$. Furthermore if (T,f) and (S,f) are weakly compatible, then ξ is a unique common fixed point of T,S and f.

Olalelu [12] proved also that Theorem 1.3 holds with the set $\mathrm{B}(T,S,f,x,y)$ instead of $\mathrm{A}(T,S,f,x,y)$.

Throughout the paper $\operatorname{co} A$ stands for the convex hull of a set A in a real vector space Y.

In 2013, Ding et al. [8] obtained the following theorem.

Theorem 1.4 (Ding et al. [8]). Let (X,d) be a complete cone metric space over a solid Hausdorff topological vector space $(Y,\preceq)$, and let T,S be selfmappings of X. Suppose that

(1) $$d(Tx,Ty)\preceq\lambda\operatorname{co}\mathrm{A}(T,S,I,x,y)$$

where $\lambda\in[0,1)$. Then T and S have a unique common fixed point in X.

In this paper, we establish a convergence theorem with error estimates for the Jungck iteration process for approximating points of coincidence and common fixed points in cone metric spaces. Our result unifies, improves and complements with error estimates the above mentioned results of Olaleru [12] and Ding et al. [8] as well as the results given in [1], [2], [3], [4], [5], [6], [7], [11], [15] and [16]. Finally, we note that the main result of the paper is new even in the classical metric space setting.

Throughout this paper we use the terminology from [13] (see also [14, Section 2]).

2. MAIN RESULT

Theorem 2.1. Let (X,d) be a cone metric space over a solid vector space $(Y,\preceq)$, and let T,S and f be three selfmappings of X such that $T(X)\cup S(X)\subset f(X)$. Suppose the following conditions hold:

(a) $d(Tx,Ty)\preceq\lambda\operatorname{co}\mathrm{A}(T,S,f,x,y)$ for all $x,y\in X$ with $x\neq y$, where $\lambda\in[0,1)$ is a constant.

(b) $d(Tx,Sx)\neq d(fx,Tx)+d(fx,Sx)$ for every $x\in X$ whenever $Tx\neq Sx$.

(c) $f(X)$ or $T(X)\cup S(X)$ is a complete subspace of X.

Then the following statements hold true:

(i) The mappings T,S,f have a unique point of coincidence $\xi\in X$.

(ii) Every (T,S)-f-sequence (fx_n) in the space X converges to ξ.

(iii) For every $n\geq 0$ the following a priori error estimate holds

$$d(fx_n,\xi)\preceq\frac{\lambda^n}{1-\lambda}d(fx_0,Tx_0)\ .$$

(iv) For every $n\geq 0$ the following two a posteriori error estimates hold:

$$d(fx_n,\xi)\preceq\frac{1}{1-\lambda}d(fx_n,fx_{n+1}),$$

$$d(fx_{n+1},\xi)\preceq\frac{\lambda}{1-\lambda}d(fx_n,fx_{n+1}).$$

(v) If the pairs (T,f) and (S,f) are weakly compatible, then ξ is a unique common fixed point of T,S,f.

Remark 2.2. It can be proved that if condition (a) of Theorem 2.1 is satisfied for all $x,y\in X$, then assumption (b) can be dropped.

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3. AUXILIARY RESULTS

In this section, we give some lemmas which will be used in the proof of the main result.

Lemma 3.1 [14]. Let $(Y,\preceq)$ be an ordered vector space. Suppose that $x,y,x_1,\ldots,x_n,y_1,\ldots,y_m$ are nonnegative vectors in Y and $0\le\lambda<1$. Then:

(i) $x\preceq \mathrm{co}\{x_1,\ldots,x_n,y\}$, $y\preceq \mathrm{co}\{y_1,\ldots,y_m\}$ $\Rightarrow$ $x\preceq \mathrm{co}\{x_1,\ldots,x_n,y_1,\ldots,y_m\}$;

ii) $x\preceq \mathrm{co}\{\lambda x,x_1,\ldots,x_n\} \Leftrightarrow x\preceq \mathrm{co}\{x_1,\ldots,x_n\}$;

ii) $x\preceq \mathrm{co}\{x_1,\ldots,x_n,y\} \Leftrightarrow x\preceq \mathrm{co}\{x_1,\ldots,x_n\}$ if $y=x_i$ for some i.

Lemma 3.2 [13]. Let (X,d) be a cone metric space over a solid vector space $(Y,\preceq)$, and let (x_n) be an nfinite sequence in X satisfying

$d(x_{n+1},x_{n+2})\preceq\lambda\, d(x_n,x_{n+1})$ for every $n\ge 0$,

where $\lambda\in[0,1)$. Then (x_n) is a Cauchy sequence. Moreover if (x_n) converges to a point $\xi\in X$, then for all $n\ge 0$ the following error estimates hold:

$$d(x_n,\xi)\preceq\frac{\lambda^n}{1-\lambda}\,d(x_0,x_1);$$

$$d(x_n,\xi)\preceq\frac{1}{1-\lambda}\,d(x_n,x_{n+1});$$

$$d(x_n,\xi)\preceq\frac{\lambda}{1-\lambda}\,d(x_n,x_{n-1}).$$

Lemma 3.3. Let (X,d) be a cone metric space over a solid vector space $(Y,\preceq)$, and let T,S and f be selfmappings of X satisfying conditions

(a) $d(fx,Tx)\preceq a\,d(fx,fy)+b\,d(fx,Sy)$ for all $x,y\in X$ with $x\ne y$,

(b) $d(fx,Sx)\preceq a\,d(fx,fy)+b\,d(fx,Ty)$ for all $x,y\in X$ with $x\ne y$,

(c) $d(Tx,Sx)\ne d(fx,Tx)+d(fx,Sx)$ for every point $x\in X$ whenever $Tx\ne Sx$,

where a and b are nonnegative constants. If there exists a (T,S)-f-sequence in X converging to a point $\xi\in f(X)$, then ξ is a point of coincidence of T,S,f.

Proof. Let (fx_n) be a (T,S)-f-sequence in X converging to $\xi\in f(X)$. By the assumption, there exists $x\in X$ such that $\xi=fx$. To prove that ξ is a point of coincidence of T,S and f it is sufficient to prove that x is a coincidence point of T,S and f. Let us consider three cases.

Case 1. Suppose there are infinitely many odd numbers n such that $x_n\ne x$. By the condition (a) with $y=x_n$ for every such n, we obtain

$$\begin{aligned} d(fx,Tx) &\preceq a\,d(fx,fx_n)+b\,d(fx,Sx_n)\\ &= a\,d(fx,fx_n)+b\,d(fx,fx_{n+1}). \end{aligned}$$

From this inequality, taking into account that $fx_n\to fx$, we conclude that $d(fx,Tx)\prec c$ for every $c\succ 0$. Hence, $d(fx,Tx)=0$ which implies $fx=Tx$. It remains to prove that $Tx=Sx$. Now let us assume that $Tx\ne Sx$. It follows from condition (c) that

$$d(Tx,Sx)\ne d(fx,Tx)+d(fx,Sx),$$

which implies that $d(Tx,Sx)\ne d(Tx,Sx)$. This contradiction proves that $Tx=Sx$. Therefore, x is a coincidence point of T,S and f.

Case 2. Analogously, if there are infinitely many even numbers n such that $x_n\ne x$, then from condition (b) we obtain that x is a coincidence point of T,S and f.

Case 3. Suppose that $x_n=x$ for all but finitely many n. Then we can choose an even number n such that $x_n=x_{n+1}=x_{n+2}=x$. From the definition of the sequence (fx_n), we have $fx_{n+1}=Tx_n$ and $fx_{n+2}=Sx_{n+1}$. In other words we have $fx=Tx$ and $fx=Sx$, which completes the proof. □

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Lemma 3.4. Let (X,d) be a cone metric space over a solid vector space $(Y,\preceq)$, and let let T,S and f be selfmappings of X. Suppose that there exists $\lambda\in[0,1)$ such that for all $x,y\in X$ with $x\neq y$,

$$d(Tx,Sy)\preceq\lambda\operatorname{co}M\{T,S,f,x,y\}.$$

Then for all $x,y\in X$ with $x\neq y$ the following inequalities hold:

$$d(fx,Tx)\preceq a\,d(fx,fy)+b\,d(fx,Sy),$$
$$d(fx,Sx)\preceq a\,d(fx,fy)+b\,d(fx,Ty),$$

where $a=\lambda/(1-\lambda)$ and $b=(1+\lambda)/(1-\lambda)$.

Lemma 3.5. Under the assumptions of Lemma 3.2, the mappings T,S and f have at most one point of coincidence in X.

Lemma 3.6 [1]. Let X be a nonempty set and T,S,f be selfmapings of X that have a unique point of coincidence $\xi\in X$. If (T,f) and (S,f) are weakly compatible, then ξ is a unique common fixed point of T,S,f.

4. PROOF OF THE MAIN RESULT

We divide the proof into two steps.

Step 1. First we shall show that every (T,S)-f-sequence (fx_n) satisfies

(2) $$d(fx_{n+1},fx_{n+2})\preceq\lambda\,d(fx_n,fx_{n+1})$$

for every $n\geq 0$. We consider the following two cases.

Case 1. Suppose $x_n\neq x_{n+1}$ for some $n\geq 0$. If n is an even number, then

$$d(fx_{n+1},fx_{n+2})=d(Tx_n,Sx_{n+1}).$$

Applying condition (a) to the right-hand side of this equality and taking into account claims (ii) and (iii) of Lemma 3.1, we get

(3) $$d(fx_{n+1},fx_{n+2})\preceq \lambda\operatorname{co}\left\{d(fx_n,fx_{n+1}),\frac{d(fx_n,fx_{n+2})}{2}\right\}.$$

Analogously, if n is an odd number, then

$$d(fx_{n+1},fx_{n+2})=d(Sx_n,Tx_{n+1}).$$

Applying condition (a) to the right-hand side of this inequality and taking into account properties (ii) and (iii) of Lemma 3.1 we again get (3). It is easy to prove that

(4) $$d(fx_n,fx_{n+2})\preceq 2\operatorname{co}\{d(fx_n,fx_{n+1}),d(fx_{n+1},fx_{n+2})\}.$$

From (3), (4) and claims (i) and (iii) of Lemma 3.1, we obtain

$$d(fx_{n+1},fx_{n+2})\preceq \lambda\operatorname{co}\{d(fx_n,fx_{n+1}),d(fx_{n+1},fx_{n+2})\}.$$

From this and Lemma 3.1 (ii), we get (2).

Case 2. Let $x_n=x_{n+1}$ for some $n\geq 0$. Therefore, to prove (1) it is sufficient to show that

(5) $$fx_{n+2}=fx_{n+1}.$$

Assume the contrary: $fx_{n+2}\neq fx_{n+1}$. If n is even, then we have that $fx_{n+1}=Tx_n$ and $fx_{n+2}=Sx_{n+1}=Sx_n$. Hence $Tx_n\neq Sx_n$. Then (b) implies that

$$d(Tx_n,Sx_n)\neq d(fx_n,Tx_n)+d(fx_n,Sx_n).$$

It is easy to see that $fx_n=Tx_n$. Then the last inequality becomes $d(Tx_n,Sx_n)\neq d(Tx_n,Sx_n)$ which is a contradiction. This proves (5) in the case when n is even. The case when n is odd can be proved similarly.

Step 2. It follows from (2) and Lemma 3.2 that (fx_n) is a Cauchy sequence. From assumption (c), it is easy to see that (fx_n) converges to a point $\xi\in f(X)$. This completes the proof of (ii). Now (iii), (iv) and (v) follow immediately from Lemma 3.2.

It follows from Lemma 3.4 that all assumptions of Lemma 3.3 are satisfied. Now it follows from Lemma 3.3 that ξ is a point of coincidence of T,S and f. From Lemma 3.5 we conclude that ξ is unique. This proves (i).

If (T,f) and (S,f) are weakly compatible, then from Lemma 3.6 follows that ξ is a unique fixed point of T,S and f. This proves (vi). □

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Example 5.1. In this example, we construct mappings T, S, f which satisfy the conditions of Theorem 2.1, but do not satisfy condition (a) of Theorem 1.3.

Let $\mathbb{R}^2$ be endowed with the pointwise ordering $\preceq$. Let $X = (a, b, c, h)$ and the mapping $d : X \times X \to \mathbb{R}^2$ be defined by

$$d(a,b) = d(a,c) = (5,2),$$
$$d(a,h) = d(b,c) = (3,3),$$
$$d(b,h) = d(c,h) = (2,5),$$

$d(x, y) = d(y, x)$ and $d(x, x) = 0$ for all points $x, y \in X$. Then (X, d) is a complete cone metric space over $\mathbb{R}^2$. Consider now the selfmappings T, S, f of X defined by

$$T = S = \begin{pmatrix} a & b & c & h \\ a & a & h & a \end{pmatrix}, \quad f = \begin{pmatrix} a & b & c & h \\ a & a & b & a \end{pmatrix}.$$

It is easy to show that the conditions of Theorem 2.1 are satisfied. In particular, condition (a) holds with $\lambda = 6/7$. On the other hand, the mappings T, S, f do not satisfy condition (a) of Theorem 1.3. Indeed, it is easy to verify that

(6) $\quad d(Ta, Tc) \preceq \lambda\, \mathrm{A}(T, S, f, a, \mathrm{c})$

is satisfied if and only if $\lambda \geq 6/5$.

Note that the last claim holds also for the inequality $d(Ta, Tc) \preceq \lambda\, \mathrm{B}(T, S, f, a, \mathrm{c})$ instead of (6).

Example 5.2. In this example we construct two mappings T, S which satisfy the conditions of Theorem 2.1 (with $f = I$), but do not satisfy condition (1) of Theorem 1.4.

Let $X = (a, b, c, h)$ and $d : X \times X \to \mathbb{R}$ be a mapping defined by

$$d(a,b) = 2, \; d(a,c) = d(a,h) = 4,$$
$$d(b,c) = d(b,h) = d(c,h) = 5,$$

$d(x, y) = d(y, x)$ and $d(x, x) = 0$ for all points $x, y \in X$. Then (X, d) is a complete metric space. Define the selfmappings T, S of X by

$$T = \begin{pmatrix} a & b & c & h \\ a & a & h & a \end{pmatrix}, \quad S = \begin{pmatrix} a & b & c & h \\ a & a & b & a \end{pmatrix}.$$

Condition (a) holds with $\lambda = 4/5$. Condition (b) also holds since $Tx \neq Sx$ is fulfilled only for $x = b$. On the other hand, condition (1) of Theorem 1.4 is not satisfied for $x = y = b$, since it is equivalent to $\lambda \geq 1$.

REFERENCE

[1] M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl., 341 (2008), 416-420. | [2] A. Azam, M. Arshad and I. Beg, Common fixed points of two maps in cone metric spaces, Rendiconti Circ. Mat. Palermo, 57 (2008), 433-441. | [3] A. Azam, M. Arshad and I. Beg, Common fixed point theorems in cone metric spaces, J. Nonlinear Sci. Appl., 2 (2009), 204-213. | [4] M. Arshad, A. Azam and P. Vetro, Some common fixed point results in cone metric spaces, Fixed Point Theory Appl., 2009 (2009), Article ID 493965. | [5] M. Abbas and B.E. Rhoades, Fixed and periodic point results in cone metric spaces, Appl. Math. Lett., 22 (2009), 511-515. | [6] I. Beg, A. Azam and M. Arshad, Some common fixed point results in cone metric spaces, Int. J. Math. Sci., 2009 (2009), Article ID 560264. | [7] V. Berinde, Common fixed points of noncommuting mappings without continuity in cone metric spaces, Taiwanese J. Math., 14 (2010), 1763-1776. | [8] H. Ding, M. Jovanović, Z. Kadelburg and S. Radenović, Common fixed point results for generalized quasicontractions on tvs-cone metric spaces, J. Comput. Anal. Appl. 15 (2013), 463-470. | [9] G. Jungck, Commuting mappings and fixed points, Amer. Math. Monthly, 83 (1976), 261-263. | [10] G. Jungck, Common fixed points for noncontinuous nonself maps on nonmetric spaces, Far East J. Math. Sci., 4 (1996), 199-215. | [11] G. Jungck, S. Radenović, S. Radojević and V. Rakočević, Common fixed point theorems for weakly compatible pairs on cone metric spaces, Fixed Point Theory Appl., 2009 (2009), Article ID 643840. | [12] J.O. Olaleru, Common fixed points of three self-mappings in cone metric spaces, Applied Mathematics E-Notes, 11 (2011), 41-49. | [13] P.D. Proinov, A unified theory for cone metric spaces and its applications to the fixed point theory, Fixed Point Theory Appl., 2013, Article ID 103. | [14] P.D. Proinov and I.A. Nikolova, Iterative approximation of fixed points of quasi-contraction mappings in cone metric spaces, J. Inequal. Appl., 2014, Article ID 226. | [15] P. Vetro, Common fixed points in cone metric spaces, Rendiconti Circ. Mat. Palermo, 56 (2007), 464-468. | [16] G. Song G., X. Sun, Y. Zhao and G. Wang , New common fixed point theorems for maps in cone metric space, Appl. Math. Lett, 23 (2010), 1033-1037. | [17] X. Zhang H. Fixed point theorem of generalized quasi-contractive mapping in cone metric space, Comput. Math. Appl., 62 (2011), 1627-1633. | |

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