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On the Ternary Quadratic Diophantine Equation $(2k+2)(x^2+y^2)-(4k-1)xy=8(k+1)z^2$

KEYWORDS: Ternary quadratic equation, Integral solutions, Special polygonal numbers, Pyramidal numbers.

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ABSTRACT *The ternary quadratic Diophantine equation represented by $(2k+2)(x^2+y^2)-(4k-1)xy=8(k+1)z^2$ is analyzed for its patterns of non – zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.*

INTRODUCTION

Ternary quadratic equations are rich in variety [1-3].For an extensive review of sizable literature and various problems, one may refer [4-19]. In this communication, we consider yet another interesting ternary quadratic equation $(2k+2)(x^2+y^2)-(4k-1)xy=8(k+1)z^2$ and obtain infinitely many non-trivial integral solutions. A few interesting relations between the solutions, special Polygonal numbers, Pyramidal numbers, Jacobsthal number and Jacobsthal-Lucas numbers [1,2] are presented.

NOTATIONS USED

- $t_{m,n}$ - Polygonal number of rank n with size m .
- P_n^m - Pyramidal number of rank n with size m .
- Pr_n - Pronic number of rank n.
- J_n - Jacobsthal number of rank n .
- j_n - Jacobsthal-Lucas number of rank n .
- S_n - Star number of rank n.

METHOD OF ANALYSIS

The Quadratic Diophantine equation with three unknowns to be solved for its non-zero distinct integral solutions is

$$(2k+2)(x^2+y^2)-(4k-1)xy=8(k+1)z^2 \quad (1$$

On substituting the linear transformations

$$x=u+v,\ y=u-v \quad (2)$$

in (1), it leads to

$$5u^2+(8k+3)v^2=(8k+8)z^2 \quad (3)$$

We obtain different patterns of integral solutions to (1) through solving (3) which are illustrated as follows:

PATTERN –I

Assume $z=5a^2+(8k+3)b^2 \quad (4)$

Write (8k+8) as

$$8k+8=(\sqrt{5}+i\sqrt{8k+3})(\sqrt{5}-i\sqrt{8k+3}) \quad (5)$$

Using (4)and(5) in (3) and employing factorization it is written as

$$(\sqrt{5}u+i\sqrt{8k+3}v)(\sqrt{5}u-i\sqrt{8k+3}v)=$$
$$\begin{cases}(\sqrt{5}+i\sqrt{8k+3})(\sqrt{5}-i\sqrt{8k+3})\\ (\sqrt{5}a+i\sqrt{8k+3}b)^2(\sqrt{5}a-i\sqrt{8k+3}b)^2\end{cases}$$

which is equivalent to the system of equations

$$(\sqrt{5}u+i\sqrt{8k+3}v)=(\sqrt{5}+i\sqrt{8k+3})(\sqrt{5}a+i\sqrt{8k+3}b)^2$$

$$(\sqrt{5}u-i\sqrt{8k+3}v)=(\sqrt{5}-i\sqrt{8k+3})(\sqrt{5}a-i\sqrt{8k+3}b)^2$$

Equating the real and imaginary parts in either of the above equations we get

$$u=5a^2-(8k+3)b^2-2ab(8k+3)$$
$$v=5a^2-(8k+3)b^2+10ab$$

In view of (2), the non-zero distinct integral solutions of (1) are

$$x=10a^2-(16k+6)b^2-2ab(8k-2)$$
$$y=-2(8k+8)ab$$

along with (4).

PROPERTIES:

1. $3[2z(a,a)-x(a,a)]$ is a nasty number.
2. $2z(a,a+1,k)+x(a,a+1,k)-10t_{4,a}+(16k-4)Pr_a=0$

3.
$x\left[(b+1)(b+2),b,k\right]-y\left[(b+1)(b+2),b,k\right]-2z\left[(b+1)(b+2),b,k\right]-120Pb^3+(32k+12)t_{4,b}=0$

4. $x\left(2^n-1,2^n+1,k\right)+y\left(2^n-1,2^n+1,k\right)+z\left(2^n-1,2^n+1,k\right)+120kJ_{2n}=0$

PATTERN –II

Introducing the linear transformations

$$u=X+(8k+3)T, v=X-5T \qquad (6)$$

in (3), it leads to

$$X^2+(40k+15)T^2=z^2 \qquad (7)$$

which is satisfied by

$$\begin{aligned} T&=2pq \\ X&=(40k+15)p^2-q^2 \qquad (8) \\ z&=(40k+15)p^2+q^2 \end{aligned}$$

From (8),(6)and (2), we get the non-zero distinct integer solutions to (1) as given below:

$$x=(80k+30)p^2-2q^2+(16k-4)pq$$
$$y=(16k+16)pq$$
$$z=(40k+15)p^2+q^2$$

PROPERTIES:

1.
$x\left[(q+1)(q+2),q,k\right]-2z\left[(q+1)(q+2),q,k\right]-6(16k-4)P_q^3+4t_{4,q}=0$

2. Each of the following expressions represents the perfect square:

a.
$2z(q-1,6q,k)-x(q-1,6q,k)+(16k-4)(S_q-1)$

b.
$2z(2^{2n},1,k)-x(2^{2n},1,k)+(16k-4)(j_{2n}-1)$

3.
$z(2^n-1,2^n+1,k)-y(2^n-1,2^n+1,k)-72kJ_{2n}=0$

4.
$x(p,p,k)-y(p,p,k)\equiv 0(\mathrm{mod}(k+1))$

PATTERN-III

Rewrite (3) as

$$5(u^2-v^2)=8(k+1)(z^2-v^2) \qquad (9)$$

which is written in the form of ratio as

$$\frac{5(u+v)}{4(z-v)}=\frac{2(k+1)(z+v)}{u-v}=\frac{\alpha}{\beta}, \beta\neq 0$$

The above equation is equivalent to the system of equations,

$$5\beta u+(4\alpha+5\beta)v-4\alpha z=0$$

$$\alpha u-[\alpha+2\beta(k+1)]v-2\beta(k+1)z=0$$

Applying the method of cross multiplication, we obtain

$$u=-4\alpha^2-10(k+1)\beta^2-16\alpha\beta(k+1)$$
$$v=-4\alpha^2+10(k+1)\beta^2$$

$$z=-4\alpha^2-10(k+1)\beta^2-10\alpha\beta \qquad (10)$$

Substituting the above values of u and v in (2), we get the non-zero distinct integral solutions to (1) to be

$$x=-8\alpha^2-16(k+1)\alpha\beta$$
$$y=-20(k+1)\beta^2-16(k+1)\alpha\beta$$

along with (10).

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PROPERTIES:

1. $2z[\alpha,\alpha(\alpha+1),k]-y[\alpha,\alpha(\alpha+1),k]-x[\alpha,\alpha(\alpha+1),k]-(64k+24)P_{\alpha}^{3}=0$
2. $2z[\alpha,(\alpha+1)(\alpha+2),k]-y[\alpha,(\alpha+1)(\alpha+2),k]-x[\alpha,(\alpha+1)(\alpha+2),k]-6(32k+12)P_{\alpha}^{3}=0$
3. $x[6\alpha(\alpha-1),1,k]-2z[6\alpha(\alpha-1),1,k]-20(k+1)-(4-16k)(S_{\alpha}-1)=0$
4. $x(3\alpha^{2},3\alpha^{2},k)-y(3\alpha^{2},3\alpha^{2},k)-9\alpha^{4}(20k-3)$ is a bi quadratic integer.
5. $2z(2^{n}-1,2^{n}+1,k)-y(2^{n}-1,2^{n}+1,k)-x(2^{n}-1,2^{n}+1,k)-3(32k+12)J_{2n}=0$

NOTE:

It is worth mentioning here that (9) may also be expressed in the form of ratios as follows:

Choice 1:

$$\frac{(u-v)}{4(z-v)(k+1)}=\frac{2(z+v)}{5(u+v)}=\frac{\alpha}{\beta},\beta\neq 0$$

Choice 2:

$$\frac{5(u-v)}{2(z+v)(k+1)}=\frac{4(z-v)}{(u+v)}=\frac{\alpha}{\beta},\beta\neq 0$$

Choice 3:

$$\frac{(u+v)}{(z-v)(k+1)}=\frac{8(z+v)}{5(u-v)}=\frac{\alpha}{\beta},\beta\neq 0$$

Choice 4:

$$\frac{5(u+v)}{8(z-v)}=\frac{(k+1)(z+v)}{(u-v)}=\frac{\alpha}{\beta},\beta\neq 0$$

By following the process mentioned in Pattern-III , the corresponding integer solutions to (1) obtained from the above choices are represented below:

Solutions for choice 1:

$$x=4\beta^{2}-16(k+1)\alpha\beta$$
$$y=40(k+1)\alpha^{2}-16(k+1)\alpha\beta$$
$$z=-2\beta^{2}-20(k+1)\alpha^{2}+10\alpha\beta$$

Solutions for choice 2:

$$x=40\beta^{2}+16(k+1)\alpha\beta$$
$$y=4(k+1)\alpha^{2}+16(k+1)\alpha\beta$$
$$z=20\beta^{2}+2(k+1)\alpha^{2}+10\alpha\beta$$

Solutions for choice 3:

$$x=-10(k+1)\alpha^{2}-16(k+1)\alpha\beta$$
$$y=-16\beta^{2}-16(k+1)\alpha\beta$$
$$z=-8\beta^{2}-5(k+1)\alpha^{2}-10\alpha\beta$$

Solutions for choice 4:

$$x=-16\alpha^{2}-16(k+1)\alpha\beta$$
$$y=-10(k+1)\beta^{2}-16(k+1)\alpha\beta$$
$$z=-5(k+1)\beta^{2}-8\alpha^{2}-10\alpha\beta$$

Conclusion:

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic Diophantine equation represented by $(2k+2)(x^{2}+y^{2})-(4k-1)xy=8(k+1)z^{2}$

As quadratic equations are rich in variety, one may search for their choices of quadratic equation with variables greater than or equal to 3 and determine their properties through special numbers.

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