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Weighted Generalized Inverse of Partitioned Matrices in Pseudo Banachiewicz - schur form

KEYWORDS: pseudo Banachiewicz- Schur form, weighted Moore-Penrose invers, weighted Drazin inverse, Pseudo-Schur complement.

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ABSTRACT *In this paper the conditions under which the weighted generalized inverses A(1,3M), A(1,4N), and can be expressed in pseudo Banachiewicz - Schur form are considered and some interesting results are established. AMS classification: 15A09, 15A15, 15A57*

Introduction

Let $C^{n\times m}$ denote the set of all complexnxmmatrices. $I_n$denotes the unit matrix of order n. By $A^* \in C^{m\times n}$ we denote the conjugate transpose matrix of $A \in C^{n\times m}$. Let us recall that the Moore-Penrose inverse of $A \in C^{n\times m}$ is the unique matrix $A^\dagger \in C^{m\times n}$ which satisfies $AA^\dagger A = A$,

$$A^\dagger AA^\dagger = A^\dagger, \left(AA^\dagger\right)^* = AA^\dagger,$$
$$\left(A^\dagger A\right)^* = A^\dagger A.$$

The Drazin inverse of $A \in C^{n\times n}$ is the matrix $A^D \in C^{n\times n}$ which satisfies $A^{k+1}X = A^k,\ XAX = X,\ AX = XA,$ for some nonnegative integer k. The least k is the index of A, denoted by $ind(A)$. Generalizing the Moore-Penrose and the Drazin inverse, the weighted Moore-Penrose inverse and the weighted Drazin inverse are defined as follows:

In this paper we have extended the results of Banachiewicz-schur form of J.K.Baksalary and G.P.Styan**[2]** our work, PseduoBanachiewicz-schur form is an extension of the above mentioned paper.

Definition 1.1

Let $A \in C^{n\times m}$ and let $M \in C^{m\times n}$ and $N \in C^{m\times m}$ be positive definite. The unique matrix $X \in C^{m\times n}$ which satisfies

$AXA = A,\ XAX = X, (MAX)^* = MAX, (NXA)^* = NXA$ **(1)**

is called the weighted Moore-Penrose inverse of A and it is denoted by $A^{\dagger}_{M,N}$.

Definition 1.2

If $A \in C^{n\times m}$ and $W \in C^{m\times n}$ are complex matrices, then the unique solution $X \in C^{n\times m}$ of the equations

$(AW)^{k+1}XW = (AW)^k$, $XWAWX = X$,

$AWX = XWA$ **(2)**

where $k=ind(AW)$, is called the W-weighted Drazin inverse of A and it is denoted by $A^{d,w}$.

Obviously for $M = I_n$ and $N = I_m$ the weighted Moore-Penrose inverse of A is the Moore-Penrose inverse of A. If $m = n$ and $W = I_n$, thenmatrix X which satisfies **(2)** is the Drazin inverse of A. It is well-knownthat

$A^{\dagger}_{M,N} = N^{-1/2}\left(M^{1/2}AN^{-1/2}\right)^{\dagger} M^{1/2}$ and $A^{d,w}$= $[(AW)^D]^2A$. Someinteresting properties of weighted Moore-Penrose and the weighted Drazininverse, among other papers, are investigated in **[9], [13]**.

For $BKN \in C^{n\times m}$, the set of inner, outer, least-squares weighted generalized and minimum-norm weighted generalized inverses, respectively are given by:

$$BKN\{1\} = \left\{X \in C^{m\times n} : [BKN]X[BKN] = [BKN]\right\},$$

$$BKN\{2\} = \left\{X \in C^{m\times n} : X[BKN]X = X\right\},$$

$$BKN\{1,3(M)\} = \left\{X \in C^{m\times n} : [BKN]X[BKN] = [BKN], \left(M[BKN]X\right)^* = M[BKN]X\right\},$$

$$BKN\{1,4(N)\} = \left\{X \in C^{m\times n} : [BKN]X[BKN] = [BKN], \left(NX[BKN]\right)^* = NX[BKN]\right\},$$

where $M \in C^{n\times n}$ and $N \in C^{m\times m}$ are positive definite matrices.

In this paper we consider matrix $BKN \in C^{(m+p)\times(n+q)}$ partitioned as

$$K = \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{bmatrix} B$$

$$= \begin{bmatrix} [K/N_{11}] & [K/N_{12}] & [K/N_{13}] \\ [K/N_{21}] & [K/N_{22}] & [K/N_{23}] \\ [K/N_{31}] & [K/N_{32}] & [K/N_{33}] \end{bmatrix}$$

$$M = \begin{bmatrix} [B/[K/N_{11}]] & [B/[K/N_{12}]] & [B/[K/N_{13}]] \\ [B/[K/N_{21}]] & [B/[K/N_{22}]] & [B/[K/N_{23}]] \\ [B/[K/N_{31}]] & [B/[K/N_{32}]] & [B/[K/N_{33}]] \end{bmatrix}$$

$$BKN_{11} = \left[B/[K/N_{11}]\right],$$

$$BKN_{12} = \left[[B/[K/N_{12}]][B/[K/N_{13}]]\right]$$

$$BKN_{21} = \begin{bmatrix} [B/[K/N_{21}]] \\ [B/[K/N_{31}]] \end{bmatrix},$$

$$BKN_{22}=\begin{bmatrix}[B/[K/N_{22}]] & [B/[K/N_{23}]]\\ [B/[K/N_{32}]] & [B/[K/N_{33}]]\end{bmatrix}$$

$$BKN=\begin{bmatrix}BKN_{11} & BKN_{12}\\ BKN_{21} & BKN_{22}\end{bmatrix}$$

Where $BKN_{11}\in C_{n\times n}$ and $BKN_{12}\in C_{n\times m}$, $BKN_{21}\in C_{m\times n}$, $BKN_{22}\in C_{m\times m}$ we use the following definition of the generalized pseudo- schur complement.

Definition 1.3

For a matrix $BKN\in C^{(m+p)\times(n+q)}$ given by **(3)**the generalized pseudo- Schur complement of *BKN* in symbol $[BBKN/KN_{11}]$, is defined by

$[BKN/BKN_{11}]=BKN_{22}-BKN_{21}[BKN_{11}]^{\beta}BK$ BKN_{12}

$$[BKN/BKN_{11}]=\begin{bmatrix}[B/K/N_{22}] & [B/K/N_{23}]\\ [B/K/N_{32}] & [B/K/N_{33}]\end{bmatrix}-\begin{bmatrix}[B/K/N_{21}]\\ [B/K/N_{31}]\end{bmatrix}[B/K/N_{11}]^{\beta}[[B/K/N_{12}][B/K/N_{13}]] \quad \textbf{(4)}$$

Where $[BKN_{11}]^{\beta}\in BKN_{11}\{1\}$.

The case BKN_{11}^{-1} instead of $[BKN_{11}]^{\beta}$, under assumption that $[BKN_{11}]^{\beta}$ is invertible, was first used by Schur**[14]**. The idea of Schur complements goes back to Sylvester (1851) and the term Schur complements was introduced by *E*. Haynsworth**[10]**. Carlson et al. **[4]** defined the generalized Schur complement by replacing the ordinary inverse with the Moore-Penrose inverse.

The Schur complement and the generalized Schurcomplement were studied by a number of authors, including their applications in statistics, matrix theory, electrical network theory, discrete-time regulator problem, sophisticated techniques and some other fields. For interesting results concerning Schur complements see also **[1], [5], [6], [7], [8], [12]**.

Banachiewicz**[3]** expressed the inverse of a partitioned matrix in terms of Schur complement. When the partitioned matrix *A*, given by **(3)**, is nonsingular and A_{11} is also nonsingular, then $[BKN/BKN_{11}]$ is nonsingular and

$$[BKN]^{-1}=\begin{bmatrix}BKN^{-1}{}_{11}+BKN^{-1}{}_{11}BKN_{12}[BKN/BKN_{11}]^{-1}BKN_{21}BKN^{-1}{}_{11} & -BKN^{-1}{}_{11}BKN_{12}[BKN/BKN_{11}]^{-1}\\ -[BKN/BKN]^{-1}BKN_{21}BKN^{-1}{}_{11} & [BKN/BKN_{11}]^{-1}\end{bmatrix}$$

where we use of $[BKN/BKN_{11}]$.

The motivations for our research are the following:

(1) The paper of Baksalary and Styan**[2]** in which they extended the result of Marsaglia and Styan**[11]**, considering the necessary and sufficient conditions such that the outer inverses, least-squares generalized inverses and minimum norm

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generalized inverses can be represented by the Banachiewicz-Schur form;

(2) The paper of Y.Wei**[15]** in which he found the sufficient conditions for the Drazin inverse to be represented by the Banachiewicz-Schur form.

Our purpose is to generalized these results for the weighted Moore-Penrose inverse and the weighted Drazin inverse of *BKN*.

1. Results

$$X=\begin{bmatrix} BKN^{\beta}{}_{11}+BKN^{\beta}{}_{11}BKN_{12}\left[BKN/BKN_{11}\right]^{\beta}BKN_{21}BKN^{\beta}{}_{11} & -BKN^{\beta}{}_{11}BKN_{12}\left[BKN/BKN_{11}\right]^{\beta} \\ -\left[BKN/BKN\right]^{\beta}BKN_{21}BKN^{\beta}{}_{11} & \left[BKN/BKN_{11}\right]^{\beta} \end{bmatrix} \quad \textbf{(5)}$$

$BKN^{\beta}{}_{11}\in BKN_{11}\{1\}$ and the positive definite matrices $M\in C_{n\times n}$ and $N\in C_{n\times n}$ are given by

$$M=\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{bmatrix},\ N=\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22}\end{bmatrix} \quad \textbf{(6)}$$

Where $M_{11}\in C_{n\times n}$, $M_{22}\in C_{p\times p}$, $N_{11}\in C_{m\times m}$, $N_{22}\in C_{q\times q}$.

We begin with the following result of Baksalary and Styn**[2]**, adopting the following notations from **[2]**.

$$E_{BKN_{11}}=I-BKN^{\beta}{}_{11}BKN_{11},$$

$$F_{BKN_{11}}=I-BKN_{11}BKN^{\beta}{}_{11},$$

$$E_{[BKN/BKN_{11}]}=I-\left[BKN/BKN\right]^{\beta}\left[BKN/BKN\right]$$

$$F_{[BKN/BKN_{11}]}=I-\left[BKN/BKN\right]\left[BKN/BKN\right]^{\beta}$$

where $\left[BKN/BKN_{11}\right]$ is the pseudo – schur complement of BKN_{11} which is defined in **(4)** and $\left[BKN/BKN\right]^{\beta}\in C_{m\times m}$.

Theorem 2.1

Let BKN and X are given by (3) and (5). Then $X\in A$ (1) if and only if $\left[BKN/BKN_{11}\right]^{\beta}\in BKN\ \{1\}$ and

$$F_{BKN_{11}}BKN_{12}E_{[BKN/BKN_{11}]}=0,$$

$$F_{[BKN/BKN_{11}]}BKN_{21}E_{BKN_{11}}=0,$$

$$F_{BKN_{11}}BKN_{12}\left[BKN/BKN_{11}\right]^{\beta} BKN_{12}E_{BKN_{11}}=0 \quad \textbf{(7)}$$

The last three conditions being independent of the choice of $BKN^{\beta}{}_{11}\in BKN_{11}\{1\}$ and $\left[BKN/BKN_{11}\right]^{\beta}\in BKN\ \{1\}$ involved in $E_{BKN_{11}}$, $F_{BKN_{11}}$, $E_{[BKN/BKN_{11}]}$, $F_{[BKN/BKN_{11}]}$

The following theorem give the necessary and sufficient conditions such that $X\in BKN\{1,3(M)\}$ under some conditions.

Theorem 2.2

If M is a positive definite matrix given by **(6)**, such that

$$\left[M_{12}\left[BKN/BKN_{11}\right]\left[BKN/BKN_{11}\right]^{\beta}\right]^{*}$$
$$=M_{12}^{*}\left[BKN_{11}\right]\left[BKN_{11}\right]^{\beta}+M_{12}^{*}F_{BKN_{11}}BKN_{12}\left[BKN/BKN_{11}\right]^{\beta}\left[BKN_{21}\right]\left[BKN_{11}\right]^{\beta}$$

$$\left[BKN/BKN_{11}\right]^{\beta}M_{22}F_{\left[BKN/BKN_{11}\right]}=E_{\left[BKN/BKN_{11}\right]}M_{22}\left[BKN/BKN_{11}\right]^{\beta} \qquad \textbf{(8)}$$

That $X\in BKN\{1,3(M)\}$ if and only if $BKN_{11}^{\beta}\in BKN_{11}\{1,3(M_{11})\}$, $\left[BKN/BKN_{11}\right]^{\beta}\in\left[BKN/BKN_{11}\right]\{1,3(M_{22})\}$ and

$F_{BKN_{11}}BKN_{12}=0, F_{\left[BKN/BKN_{11}\right]}BKN_{21}=0$ **(9)**

The last two conditions are independent of the choice of $BKN_{11}^{\beta}\in BKN_{11}\{1\}$ and $\left[BKN/BKN_{11}\right]^{\beta}\in\left[BKN/BKN_{11}\right]\{1\}$ involved in $F_{BKN_{11}}$ and $F_{\left[BKN/BKN_{11}\right]}$.

Proof

Suppose that

$$BKN_{11}^{\beta}\in BKN_{11}\{1,3(M_{11})\},$$

$\left[BKN/BKN_{11}\right]^{\beta}\in\left[BKN/BKN_{11}\right]\{1,3(M_{22})\}$ and the conditions **(9)** are satisfied. Then the conditions from the **Theorem 2.1** are satisfied and X is an inner inverse of BKN. Also, we have that

$$\left(M\left(BKN\right)X\right)_{11}=M_{11}(BKN_{11})(BKN_{11})^{\beta}-M_{11}F_{BKN_{11}}BKN_{12}\left[BKN/BKN_{11}\right]^{\beta}BKN_{21}BKN_{11}^{\beta}$$
$$+M_{12}F_{\left[BKN/BKN_{11}\right]}BKN_{21}BKN_{11}^{\beta}$$
$$=M_{11}BKN_{11}BKN_{11}^{\beta},$$

$$\left(M\left(BKN\right)X\right)_{12}=M_{11}F_{BKN_{11}}BKN_{12}\left[BKN/BKN_{11}\right]^{\beta}+M_{12}\left[BKN/BKN_{11}\right]\left[BKN/BKN_{11}\right]^{\beta}$$
$$=M_{12}\left[BKN/BKN_{11}\right]\left[BKN/BKN_{11}\right]^{\beta}$$

$$\left(M\left(BKN\right)X\right)_{12}=M_{12}^{*}BKN_{11}BKN_{11}^{\beta}-M_{12}^{*}F_{BKN_{11}}BKN_{12}\left[BKN/BKN_{11}\right]^{\beta}BKN_{21}BKN_{11}^{\beta}$$
$$+M_{22}F_{\left[BKN/BKN_{11}\right]}BKN_{21}BKN_{11}^{\beta}$$
$$=M_{12}^{*}BKN_{11}BKN_{11}^{\beta}$$

$$\left(M\left(BKN\right)X\right)_{22}=M_{12}^{*}F_{BKN_{11}}BKN_{12}\left[BKN/BKN_{11}\right]^{\beta}+M_{22}\left[BKN/BKN_{11}\right]\left[BKN/BKN_{11}\right]^{\beta}$$
$$=M_{22}\left[BKN/BKN_{11}\right]\left[BKN/BKN_{11}\right]^{\beta}$$

Obviously,

$\left(M\left(BKN\right)X\right)_{11}=\left(M\left(BKN\right)X\right)_{11}^{*}$,

$\left(M\left(BKN\right)X\right)_{12}=\left(M\left(BKN\right)X\right)_{12}^{*}$ and

$\left(M\left(BKN\right)X\right)_{22}=\left(M\left(BKN\right)X\right)_{22}^{*}$

i.e, $\left(M\left(BKN\right)X\right)=\left(M\left(BKN\right)X\right)^{*}$

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On the other hand, let $X \in BKN\{1,3(M)\}$ then the conditions **(7)**are satisfied and $\left(M\left(BKN\right)X\right)=\left(M\left(BKN\right)X\right)^*$.

$$M_{11}F_{BKN_{11}}BKN_{12}\left[BKN/BKN_{11}\right]^{\beta}=\left[M_{22}F_{[BKN/BKN_{11}]}BKN_{21}BKN_{11}^{\beta}\right]^* \text{ and}$$

$$\left(M_{11}F_{BKN_{11}}BKN_{12}\left[BKN/BKN_{11}\right]^{\beta}\right)\left(M_{11}F_{BKN_{11}}BKN_{12}\left[BKN/BKN_{11}\right]^{\beta}\right)^* = M_{11}F_{BKN_{11}}BKN_{12}\left[BKN/BKN_{11}\right]^{\beta}$$

$$M_{22}F_{[BKN/BKN_{11}]}BKN_{21}BKN_{11}^{\beta}=M_{11}F_{BKN_{11}}BKN_{12}E_{[BKN/BKN_{11}]}M_{22}\left[BKN/BKN_{11}\right]^{\beta}$$

$$F_{[BKN/BKN_{11}]}BKN_{21}BKN_{11}^{\beta}=0$$

Hence, $M_{11}F_{BKN_{11}}BKN_{12}\left[BKN/BKN_{11}\right]^{\beta}=0$, i.e, $M_{11}F_{BKN_{11}}BKN_{12}=0$ and $M_{22}F_{[BKN/BKN_{11}]}BKN_{21}BKN_{11}^{\beta}=0$, i.e, $M_{22}F_{[BKN/BKN_{11}]}BKN_{21}=0$.

Using the fact that *M* is invertible, we have that M_{11} and M_{22} are also invertible, so we obtain the conditions **(9)**.

By the $\left(M\left(BKN\right)X\right)_{11}=\left(M\left(BKN\right)X\right)_{11}^*$ and $\left(M\left(BKN\right)X\right)_{22}=\left(M\left(BKN\right)X\right)_{22}^*$ it follows that $BKN_{11}^{\beta}\in BKN_{11}\{1,3(M_{11})\}$ and $\left[BKN/BKN_{11}\right]^{\beta}\in\{1,3(M_{22})\}$

We have the $\left(M\left(BKN\right)X\right)_{21}=\left(M\left(BKN\right)X\right)_{21}^*$, so we obtain that

The independence of the conditions **(9)** of the choice of $BKN_{11}^{\beta}\in BKN_{11}\{1\}$ in $F_{BKN_{11}}$ and $\left[BKN/BKN_{11}\right]^{\beta}\in\left[BKN/BKN_{11}\right]\{1\}$ in $F_{[BKN/BKN_{11}]}$ follows by the same arguments as in the proof of **Theorem 2.1**

Corollary 2.1

If *M* is a positive definite matrix given by **(6)**, such that

$$\left[BKN/BKN_{11}\right]^{\beta}M_{22}F_{[BKN/BKN_{11}]} = E_{[BKN/BKN_{11}]}M_{22}\left[BKN/BKN_{11}\right]^{\beta},$$

$$M_{12}^*F_{BKN_{11}}BKN_{12}=0,$$

$$M_{12}\left[BKN/BKN_{11}\right]\left[BKN/BKN_{11}\right]^{\beta} = \left[M_{12}^*BKN_{11}BKN_{11}^{\beta}\right]^*, \quad \textbf{(10)}$$

Then $X\in BKN\{1,3(M)\}$ if and only if $BKN_{11}^{\beta}\in BKN_{11}\{1,3(M_{11})\}$, $\left[BKN/BKN_{11}\right]^{\beta}\in S\{1,3(M_{22})\}$ and

$F_{BKN_{11}} BKN_{12} = 0$,

$F_{[BKN/BKN_{11}]} BKN_{21} = 0$ **(11)**

The last two conditions are independent of the choice of $BKN_{11}^{\beta} \in BKN_{11}\{1\}$ and $[BKN/BKN_{11}]^{\beta} \in BKN\{1\}$ involved in $F_{BKN_{11}}$ and $F_{[BKN/BKN_{11}]}$ notice that for$M=I_{m+p}$, the conditions **(8)** are satisfied , so we obtain the **Theorem 3** in **[2]** as a special case for $M = I_{m+p}$.

Corollary 2.2

If $BKN_{11}^{\beta} \in BKN_{11}\{1,3(M_{11})\}$,

$[BKN/BKN_{11}]^{\beta} \in [BKN/BKN_{11}]\{1,3(M_{22})\}$ and

$F_{BKN_{11}} BKN_{12} = 0$, $F_{[BKN/BKN_{11}]} BKN_{21} = 0$, **(12)**

$$M_{12}[BKN/BKN_{11}][BKN/BKN_{11}]^{\beta} = \left[M_{12}^{*} BKN_{11} BKN_{11}^{\beta}\right]^{*}$$

then $X \in BKN\{1,3(M)\}$.

The following theorem given the necessary and sufficient conditions for $X \in BKN\{1,4(N)\}$.

Theorem 2.3

If N is a positive definite matrix given by **(6)** such that

$$\left[N_{12}[BKN/BKN_{11}]^{\beta}[BKN/BKN_{11}]\right]^{*} = N_{12}^{*} BKN_{11}^{\beta} BKN_{11} + N_{12}^{*} BKN_{11}^{\beta} BKN_{12}[BKN/BKN_{11}]^{\beta} BKN_{21} E_{BKN_{11}},$$

$$[BKN/BKN_{11}]^{\beta} M_{22} F_{[BKN/BKN_{11}]} = E_{[BKN/BKN_{11}]} M_{22}[BKN/BKN_{11}]^{\beta} \quad \textbf{(13)}$$

Then $X \in A\{1,4(N)\}$ if and only if $BKN_{11}^{\beta} \in BKN_{11}\{1,4(N_{11})\}$,

$[BKN/BKN_{11}]^{\beta} \in [BKN/BKN_{11}]\{1,4(N_{22})\}$ and

$BKN_{12} E_{[BKN/BKN_{11}]} = 0, BKN_{21} E_{BKN_{11}} = 0$ **(14)**

Proof

The proof is analogous to the proof of **Theorem 2.2**

Corollary 2.3

If N is nonnegative matrix given by **(6)**, such that

$$[BKN/BKN_{11}]^{\beta} N_{22} F_{[BKN/BKN_{11}]} = E_{[BKN/BKN_{11}]} N_{22}[BKN/BKN_{11}]^{\beta},$$

$$N_{12}^{*} BKN_{11}^{\beta} BKN_{12} = 0,$$

$$N_{12}[BKN/BKN_{11}]^{\beta}[BKN/BKN_{11}] = \left[N_{12}^{*} BKN_{11}^{\beta} BKN_{11}\right]^{*} \quad \textbf{(15)}$$

Then $X \in BKN\{1,4(BKN)\}$ if and only if $BKN_{11}^{\beta} \in BKN_{11}\{1,3(N_{11})\}$,

$[BKN/BKN_{11}]^{\beta} \in [BKN/BKN_{11}]\{1,4(N_{22})\}$ and

$BKN_{12} E_{[BKN/BKN_{11}]} = 0, BKN_{21} E_{BKN_{11}} = 0$ **(16)**

Also, **Theorem 4** in **[2]** is obtained as a special case for $N=I_{n+q}$.

Corollary 2.4

If $BKN_{11}^{\beta} \in BKN_{11}\{1,3(N_{11})\}$,

$[BKN/BKN_{11}]^{\beta} \in [BKN/BKN_{11}]\{1,4(N_{22})\}$ and $A_{12}E_{[BKN/BKN_{11}]} = 0$,

$BKN_{21}E_{BKN_{11}} = 0$,

$$N_{12}[BKN/BKN_{11}]^{\beta}[BKN/BKN_{11}] = \left[N_{12}^{*}BKN_{11}^{\beta}BKN_{11}\right]^{*},$$

then $X \in BKN\{1,3(M)\}$

It is easy to se that **Theorem 2.2** and **Theorem 2.3** are satisfied if we suppose that M_{11}, M_{22}, N_{11} and N_{22} are invertible, instead of the fact that *M* and *N* are invertible matrices.

Using the results four **Theorem 2.2**, **Theorem 2.3** and **Theorem 2** in **[2]** we obtain the necessary and sufficient conditions such that the weighted Moore-Penrose inverse of BKN, $BKN_{M,N}^{\beta}$ was the PseduoBanachiewicz-Schurform where *M* and *N* are matrices which satisfy the conditions **(8)** and **(13).**

Theorem 2.4

Let *M* and *N* be the matrices which satisfy the conditions **(8)** and **(13)**. Then $X = BKN_{M,N}^{\beta}$ if and only if

$BKN_{11}^{\beta} = (BKN_{11})_{M_{11}N_{11}}^{\dagger}$,

$[BKN/BKN_{11}]^{\beta} = [BKN/BKN_{11}]_{M_{22}N_{22}}^{\dagger}$

and

$F_{BKN_{11}}BKN_{12} = 0$,

$F_{[BKN/BKN_{11}]}BKN_{21} = 0$,

$BKN_{12}E_{[BKN/BKN_{11}]} = 0$,

$$BKN_{21}E_{BKN_{11}} = 0 \qquad \textbf{(17)}$$

if $M = I_{m+p}$ and $N = I_{n+q}$, then the conditions **(8)** and **(13)** are obviously satisfied, so from **Theorem 2.4** we obtain the necessary and sufficient conditions $X = BKN^{\dagger}$ also with less restrictive conditions for the matrices *M* and *N* we obtain the sufficient conditions for $X = BKN_{M,N}^{\dagger}$.

Corollary 2.5

If $BKN_{11}^{\beta} = BKN_{M_{11}N_{11}}^{\dagger}$,

$[BKN/BKN_{11}]^{\beta} = [BKN/BKN_{11}]_{M_{22},N_{22}}^{\dagger}$

and $F_{BKN_{11}}BKN_{12} = 0$,

$F_{[BKN/BKN_{11}]}BKN_{21} = 0,$

$BKN_{21}E_{[BKN/BKN_{11}]} = 0$,

$BKN_{21}E_{BKN_{11}} = 0,$

$$M_{12}[BKN/BKN_{11}][BKN/BKN_{11}]^{\beta} = \left[M_{12}^{*}BKN_{11}BKN_{11}^{1}\right]^{*},$$

$$N_{12}[BKN/BKN_{11}]^{\beta}[BKN/BKN_{11}] = \left[N_{12}^{*}BKN_{11}^{1}BKN_{11}\right]^{*}$$

then $X = BKN_{m,n}^{\dagger}$.

It is interesting to notice that if we denote by

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$$Z=\begin{bmatrix} I & 0 \\ BKN_{21}BKN_{11}^{1} & I \end{bmatrix}$$

$$\begin{bmatrix} BKN_{11} & 0 \\ 0 & [BKN/BKN_{11}] \end{bmatrix}\begin{bmatrix} I & BKN_{11}^{1}BKN_{12} \\ 0 & I \end{bmatrix},$$

Where $BKN_{11}^{1}\in BKN_{11}\{1\}, then$

$$BKN=\begin{bmatrix} BKN_{11} & BKN_{12} \\ BKN_{21} & BKN_{22} \end{bmatrix}$$

$$=\begin{pmatrix} 0 & F_{BKN_{11}}BKN \\ BKN_{21}E_{BKN_{11}} & 0 \end{pmatrix}+Z,$$

and if the expression (5) of X is rewritten as following matrices products

$$X=\begin{pmatrix} I & -BKN_{11}^{\beta}BKN_{12} \\ 0 & I \end{pmatrix}\begin{pmatrix} BKN_{11}^{\beta} & 0 \\ 0 & (BKN/BKN_{11})^{\dagger} \end{pmatrix}\begin{pmatrix} I & 0 \\ -BKN_{21}BKN_{11}^{\beta} & I \end{pmatrix}$$

Then it is easy to see that $X\in Z\{1\}$. Moreover if the conditions $F_{BKN_{11}}BKN_{12}=0$ and $BKN_{21}E_{BKN_{11}}=0$ hold, we can obtain that $BKN=Z$ and therefore $X\in BKN\{1,2\}$ if and only if $BKN_{11}^{1}\in BKN\{1,2\}$ and $[BKN/BKN_{11}]^{\beta}\in[BKN/BKN_{11}]\{1,2\}$.

In the rest of the paper we consider the sufficient conditions such that the W-weighted drazin inverse can be represented in the pseudoBanachewieczschur form.

Recall the for an arbitrarg matrix $W\in C_{(n+q)\times(m+p)}$ there exist non singular matrix $P\in_{(n+q)\times(n+q)}$ and $Q\in C_{(m+q)\times(m+q)}$ such that

$$W=PW^{1}Q^{-1}=P\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}Q^{-1},$$

where $W_1\in C_{n\times m}, W_2\in_{q\times p}$.

Hence, if $BKN=QBKN^{1}P^{-1}$ and $X=QX^{1}P^{-1}$, then X is the W-Weighted Drazin inverse of BKN^{β} with this reason we will naturally assume that $W\in C_{(n+q)\times(m+p)}$ has the following form

$$W=\begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix},$$

$$W_1\in C_{m\times m},\ W_2\in C_{q\times q} \qquad \textbf{(18)}$$

In the next result concerning W-weighted Drazin inverse further more, we will consider matrix $BKN\in C_{(m+p)\times(n+q)}$ given by (3) modified Pseudo schur complement given by

$$[BKN/BKN_{11}]$$
$$=BKN_{22}-BKN_{21}W_1BKN_{11}^{\beta}W_1BKN_{12} \qquad \textbf{(19)}$$

And modified

$$X=\begin{bmatrix} BKN_{11}^{\beta}+BKN_{11}^{\beta}W_1BKN_{12}W_2[BKN/BKN_{11}]^{\beta}W_2BKN_{21}W_1BKN_{11}^{\beta} & -BKN_{11}^{\beta}W_1BKN_{12}W_2[BKN/BKN_{11}]^{\beta} \\ -[BKN/BKN_{11}]^{\beta}W_2BKN_{21}W_1BKN_{11}^{\beta} & [BKN/BKN_{11}]^{\beta} \end{bmatrix} \qquad \textbf{(20)}$$

Where $BKN_{11}^{\beta}=BKN_{11}^{d,w}$,

$[BKN/BKN_{11}]^{\beta}=[BKN/BKN_{11}]^{d,w_2}$.

Theorem 2.5

Let *BKN, X, W, S* be given by **(3)**, **(20), (18), (19)** respectively if

$$BKN_{12}W_2 = BKN_{11}W_1BKN_{11}^{\beta}W_1BKN_{12}W_2$$

$$= BKN_2W_2\left[BKN/BKN_{11}\right]W_2\left[BKN/BKN_{11}\right]^{\beta}W_2 \quad \textbf{(21)}$$

$$BKN_{21}W_1 = BKN_{21}W_1BKN_{11}W_1BKN_{11}^{\beta}W_1$$

$$=\left[BKN/BKN_{11}\right]W_2\left[BKN/BKN_{11}\right]^{\beta}W_2BKN_{21}W_1 \quad \textbf{(22)}$$

$$W_1BKN_{12} = W_1BKN_{12}W_2\left[BKN/BKN_{11}\right]^{\beta}W_2\left[BKN/BKN_{11}\right], \quad \textbf{(23)}$$

$$W_1BKN_{21} = W_1BKN_{21}W_1BKN_{11}^{\beta}W_1BKN_{11} \quad \textbf{(24)}$$

$$BKN_{22}W_2 = A_{22}W_2\left[BKN/BKN_{11}\right]W_2\left[BKN/BKN_{11}\right]^{\beta}W_2 \quad \textbf{(25)}$$

Then $X = BKN^{d,w}$.

Proof

By a straight forward computation, we obtain that

$$\left(BKNWX\right)_{11} = BKN_{11}W_1BKN_{11}^{\beta} - \left(BKN_{12}W_2 - BKN_{11}W_1BKN_{11}^{\beta}W_1BKN_{12}W_2\right)$$

$$\left[BKN/BKN_{11}\right]^{\beta}W_2BKN_{21}W_1BKN_{11}^{\beta}$$

$= BKN_{11}W_1BKN_{11}^{\beta}$, using the first part of **(21)**,

$$\left(BKNWX\right)_{12} = \left(BKN_{12}W_2 - BKN_{11}W_{11}BKN_{11}^{\beta}W_1BKN_{12}W_2\right)\left(BKN/BKN_{11}\right)^{\beta}$$

$= 0$ by the first the part of **(21)**,

$$\left(BKNWX\right)_{21} = BKN_{21}W_1BKN_{11}^{\beta} - \left(BKN_{22} - BKN_{21}W_1BKN_{11}^{\beta}W_1BKN_{12}\right)$$

$$W_2\left[BKN/BKN_{11}\right]^{\beta}W_2BKN_{21}W_1BKN_{11}^{\beta}$$

$$= BKN_{21}W_1BKN_{11}^{\beta} - \left[BKN/BKN_{11}\right]W_2\left[BKN/BKN_{11}\right]^{\beta}W_2BKN_{21}W_1BKN_{11}^{\beta}$$

= 0 by the second part of **(22)**,

$$\left(BKNWX\right)_{22} = \left(BKN_{22} - BKN_{21}W_1BKN_{11}^{\beta}W_1BKN_{12}\right)W_2\left[BKN/BKN_{11}\right]^{\beta}$$

$$=\left[BKN/BKN_{11}\right]W_2\left[BKN/BKN_{11}\right]^{\beta}$$

Similarly,

$$\left(XWBKN\right)_{11}$$

$$= BKN_{11}^{\beta}W_1BKN_{11} - BKN_{11}^{\beta}W_1BKN_{12}W_2\left[BKN/BKN_{11}\right]^{\beta}\left(W_2BKN_{21} - W_2BKN_{21}W_1BKN_{11}^{\beta}W_1BKN_{11}\right)$$

$= BKN_{11}^{\beta}W_1BKN_{11}$ using **(24)**,

$$\left(XWBKN\right)_{12}$$

$$= BKN_1^{\beta} W_1 BKN_{12} - BKN_{11}^{\beta} W_1 BKN_{12} W_2 \left[BKN / BKN_{11}\right]^{\beta} W_2 \left(BKN_{22} - BKN_{21} W_1 BKN_{11}^{\beta} W_1 BKN_{12}\right)$$

$= 0$, using **(23)**

$$\left(XWBKN\right)_{21} = \left[BKN / BKN_{11}\right]^{\beta} \left(W_2 BKN_{21} - W_2 BKN_{21} W_1 BKN_{11}^{\beta} W_1 BKN_{11}\right)$$

$= 0$, using **(24)**,

$$\left(XWBKN\right)_{22} = \left[BKN / BKN_{11}\right]^{\beta} W_2 \left(BKN_{22} - BKN_{21} W_1 BKN_{11}^{\beta} W_1 BKN_{12}\right)$$

$$= \left[BKN / BKN_{11}\right] W_2 \left[BKN / BKN_{11}\right]^{\beta}$$

Now, $$BKNWX = \begin{bmatrix} BKN_{11} W_{11} BKN_{11}^{\beta} & 0 \\ 0 & \left[BKN / BKN_{11}\right] W_2 \left[BKN / BKN_{11}\right]^{\beta} \end{bmatrix}$$

And $$XWBKN = \begin{bmatrix} BKN_{11}^{\beta} W_1 BKN_{11} & 0 \\ 0 & \left[BKN / BKN_{11}\right]^{\beta} W_2 \left[BKN / BKN_{11}\right] \end{bmatrix},$$

So, $BKNWX = XWBKN$.

Using the facts that $BKN_{11}^{\beta} = BKN_{11}^{d:w,}$ and $\left[BKN / BKN_{11}\right]^{\beta} = \left[BKN / BKN_{11}\right]^{d,w_2}$ we obtain that $XWBKNWX = X$. Also using **(25), (22)** and **(21)** it follows that,

$$BKNWX = \begin{bmatrix} BKN_{11} W_1 BKN_{11}^{\beta} & 0 \\ 0 & \left[BKN / BKN_{11}\right] W_2 \left[BKN / BKN_{11}\right]^{\beta} \end{bmatrix}$$

And $$XWBKN = \begin{bmatrix} BKN_{11}^{1} W_1 BKN_{11} & 0 \\ 0 & \left[BKN / BKN_{11}\right]^{1} W_2 \left[BKN / BKN_{11}\right] \end{bmatrix}$$

So $BKNWX = XWBKN$

Using the facts that $BKN_{11}^{\beta} = BKN_{11}^{d,w_1}$ and $\left[BKN / BKN_{11}\right]^{\beta} = \left[BKN / BKN_{11}\right]^{d,w_2}$ we obtain that $XWBKNWX = X$. Also, using **(25), (22)** and **(21)** it follows that.

$$\left(BKNW\right)^2 XW = \begin{bmatrix} \left(BKN_{11} W_1\right)^2 BKN_{11}^{\beta} W_1 & BKN_{12} W_2 \left[BKN / BKN_{11}\right] W_2 \left[BKN / BKN_{11}\right]^{\beta} W_2 \\ BKN_{21} W_1 BKN_{11} W_1 BKN_{11}^{\beta} W_1 & BKN_{22} W_2 \left[BKN / BKN_{11}\right] W_2 \left[BKN / BKN_{11}\right]^{\beta} W_2 \end{bmatrix}$$

$$= BKNW + \begin{bmatrix} \left(BKN_{11} W_1\right)^2 BKN_{11}^{\beta} W_1 - BKN_{11} W_1 & 0 \\ 0 & 0 \end{bmatrix}$$

By the induction, using the first part of **(22)**, we obtain that $\left(BKNW\right)^{m+1} XW = \left(BKNW\right)^{m}$, for an arbitrary $m \geq ind\left(BKN_{11}W_{1}\right)$.

From **Theorem 2.5**, we obtain the result of [wei**Theorem 1, [15]**] when $m = n$, $p = q$ and $w = I_{m+p}$

Corollary 2.6

Let BKN and X are given by **(3)** and **(5)**. If $F_{BKN_{11}} BKN_{12} = 0$, $BKN_{12} E_{[BKN/BKN_{11}]} = 0$, $BKN_{22} E_{[BKN/BKN_{11}]} = 0$ then $X = BKN^{d}$.

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