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Generalized Separable Method to solve Second order non linear Partial Differential Equations

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ABSTRACT *This paper deals with exact analytical method used in many real world problems to solve governing non linear partial differential equations. The Generalized Separable method is discussed with examples to solve second order non linear partial differential equations. This paper presents exact solutions to non linear equations which are useful in many engineering applications such as heat and mass transfer, wave theory, nonlinear mechanics, hydrodynamics, gas dynamics, plasticity theory, nonlinear acoustics, combustion theory, nonlinear optics, theoretical physics, differential geometry, control theory, chemical engineering sciences, biology, and other fields.*

Introduction

Nonlinear partial differential equations are encountered in various fields ofmathematics, physics, chemistry, and biology, and numerous applications. Exact (closed-form) solutions of differential equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science.The governing differential equations of different phenomena arising in porous media are in the form of non linear partial differential equations. It is a challenging task to solve these equations representing different phenomena. Some standard transformation like similarity transformation are used to transformnon linear partial differential equation into non linear ordinary differential equation but still it is difficult to get its exact solution. Many researchers are working on approximate solution of such non linear partial differential equation using different numerical techniques. Here our attempt is to obtain a classical exact solution of non linear partial differential equation by using generalized separable method which is described by many authors in their book such as Galaktionav and Posashkov[1989], Galaktionov[1995], Zaitsev and Polyanin [1996].

Separation of variables is the most common approach to solve linear equations of mathematical physics. For equations in two independent variables $x\&t$ and a dependent variable w, this approach involves searching for classical exact solutions in the form of the product of functions depending on different arguments:

$$w(x.t) = \varphi(x)\Psi(t) \qquad (1)$$

The integration of a few classes of first order nonlinear partial differential equations is based on searching for exact solutions in the form of the sum of functions depending on different arguments:

$$w(x.t) = \varphi(x) + \Psi(t) \qquad (2)$$

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Some second- and higher order nonlinear equations of mathematical physics also have exact solutions of the form (1) or (2). Such solutions are called multiplicative separable and additive separable, respectively.

Over the last decade, more sophisticated, generalized and functional separable solutions have been obtained for a number of second-order nonlinear equations of mathematical physics.

For example, Galaktionov and Posashkov[1989] and Galaktionov, Posashkov, and Svirshchevskii[1995] obtained generalized separable solutions with the forms $w(x,t) = \varphi(x)\Psi(t) + \aleph(t)$ and $w(x,t) = \varphi(x)\Psi(t) + \aleph(x)$ for some classes of parabolic and hyperbolic equations with quadraticnonlinearities. The results of GalaktionovandPosashkov[1994] and Galaktionov [1995] are based on finding finite-dimensional subspacesthat are invariant under appropriate nonlinear differential operators (in practice, the authors hadto find a system of coordinate functions in one of the variables by the method of undeterminedcoefficients).

In Grundland A. M., Infeld, E. A[1992], Miller J., Rubel L. A.[1993],Zhdanov R. Z[1994] and Andreev V. K. et.al [1999], all nonlinear heat (diffusion) and wave equations of the form $\partial_{xx} w \pm \partial_{yy} w = f(w)$ which admit functional separable solutions having the form $w(x.y) = f(z)$where $z = \varphi(x) + \Psi(y)$ are described. Doyle and Vassiliou [1988] indicated all one-dimensional non stationary heat equations $\partial_t w = \partial_x [f(w)\partial_x(w)]$ which admit the solutions of the form $w(x.t) = f(z)$where $z = \varphi(x) + \Psi(t)$. Many nonlinear mathematical physics equations of various types that admit generalized and functional separable solutions are described (special attention was paid to equations of general form which depend on arbitrary functions) in Zaitsev V. F., Polyanin A. D [1996], Polyanin, A. D.,Zhurov, A. I [1988], Polyanin A. D., Vyazmin A. V., Zhurov A. I., Kazenin D. A [1998] and Polyanin, A. D., Zhurov, A. I., Vyazmin, A. V. [2000].

Functional differential equations that involve unknown functions (and their derivatives) with different arguments arise when searching for ordinary, generalized, and functional separable solutions. The current supplement presents direct methods for and examples of constructing such solutions and reviews application of these methods to solving various classes of the second-, third-, fourth-, and higher-order partial differential equations (in total, about 150 nonlinear equations with solutions are described). Special attention is paid to equations of heat and mass transfer theory, wave theory, and hydrodynamics, as well as mathematical physics equations of general form that involve arbitrary functions. First time this method is used to solve nonlinear partial differential equation for different phenomena arising in fluid flow through porous media. It should be noted that often exact generalized and functional separable solutions cannot be obtained by group theoretic methods or other well-known methods.

Structure of generalized separable solution

To simplify the presentation, we confine ourselves to the case of mathematical physics equations in two independent variables *x&t* and a dependent variable *w* (one of the independent variables can play the role of time). Linear separable equations of mathematical physics admit exact solutions in the form

$$w(x.t) = \varphi_1(x)\Psi_1(t) + \varphi_2(x)\Psi_2(t) + \cdots + \varphi_n(x)\Psi_n(t) \quad (3)$$

where the $w_i = \varphi_i(x)\Psi_i(t)$) are particular solutions. Where $i = 1,2,3, \dots \dots, n$.

Many nonlinear partial differential equations with quadratic or power nonlinearities also have exact solutions of the form (3).

$$f_1(x)g_1(t)\prod_1[w] + f_2(x)g_2(t)\prod_2[w] + \cdots + f_n(x)g_n(t)\prod_n[w] = 0 \quad (4)$$

Where $\prod_i[w]$ are differential forms that are the products of nonnegative integer powers of the function *w* and its partial derivatives $w_x, w_y, w_{xx}, w_{xy}, w_{yy}, w_{xxx}\ etc.$We will refer to solutions (3) of nonlinear equations (4) as *generalized separable solutions*. In general, the functions $\varphi_i(x)\ and\ \Psi_j(t)$ in (3) are not known in advance and are to be identified.

Note that most common of the generalized separable solutions are solutions of the special form

$$w(x.t) = \varphi(x)\Psi(t) + \aleph(x) \quad (5)$$

The independent variables on the right hand side can be swapped. In the special case$\varphi(x) = 0$, this is a multiplicative separable solution, and if $\varphi(x) = 1$, this is an additive separable solution.

General form of functional differential equations

In general, on substituting expression (3) into the differential equation (4), one arrives at a functional differential equation

$$\phi_1(X)\Psi_1(T) + \phi_2(X)\Psi_2(X) + \cdots + \phi_n(X)\Psi_n(T) = 0 \quad (6)$$

$$where\ \phi_j(X) = \phi_j(x, \varphi_1, \varphi_1{}', \varphi_1{}'', \dots, \varphi_n, \varphi_n{}', \varphi_n{}'')$$

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$$and\ \Psi_j(X) = \Psi_j\left(x, \Psi_1, \Psi_1', \Psi_1'', \dots, \Psi_n, \Psi_n', \Psi_n''\right) \quad (7)$$

Here, for simplicity, the formulas are written out for the case of a second order equation (5); for higher order equations, the right hand sides of relations (3) will contain higher order derivatives of $\varphi_i(x) and \Psi_j(t)$.

The method for solving functional differential equation (6), (7) is discussed as follows:

Simplified scheme for constructing generalized separable solution

To construct exact solutions of equations (5) with quadratic or power nonlinearities that do not depend explicitly on x (all $f_i$constant), it is reasonable to use the following simplified approach. As mentioned before, we seek solutions in the form of finite sums (4) in this paper. We assume that the system of coordinate functions $\varphi_i(x)$ is governed by linear differential equations with constant coefficients. The most common solutions of such equations are of the forms

$$\varphi_i(x) = x^i, \varphi_i(x) = e^{\lambda x}, \varphi_i(x) = \sin(\alpha x), \varphi_i(x) = \cos(\beta x) \quad (8)$$

Finite chains of these functions (in various combinations) can be used to search for separablesolutions (4), where the quantities $\lambda, \alpha and \beta$ are regarded as free parameters. The other system of functions $\{\Psi_i(T)\}$ is determined by solving the nonlinear equations resulting from substituting (8) into the equation under consideration. However, specifying one of the systems of coordinate functions $\{\Psi_i(T)\}$ simplifies the procedure of finding exact solutions substantially. The drawback of this approach is that some solutions of the form (4) can be overlooked. It is significant that the overwhelming majority of generalized separable solutions known to date, for partial differential equations with quadratic nonlinearities are determined by coordinate functions (8). Hence we have considered the solutions as quadratic polynomial in *x* with coefficients as functions of *t* with physical significance in this paper.

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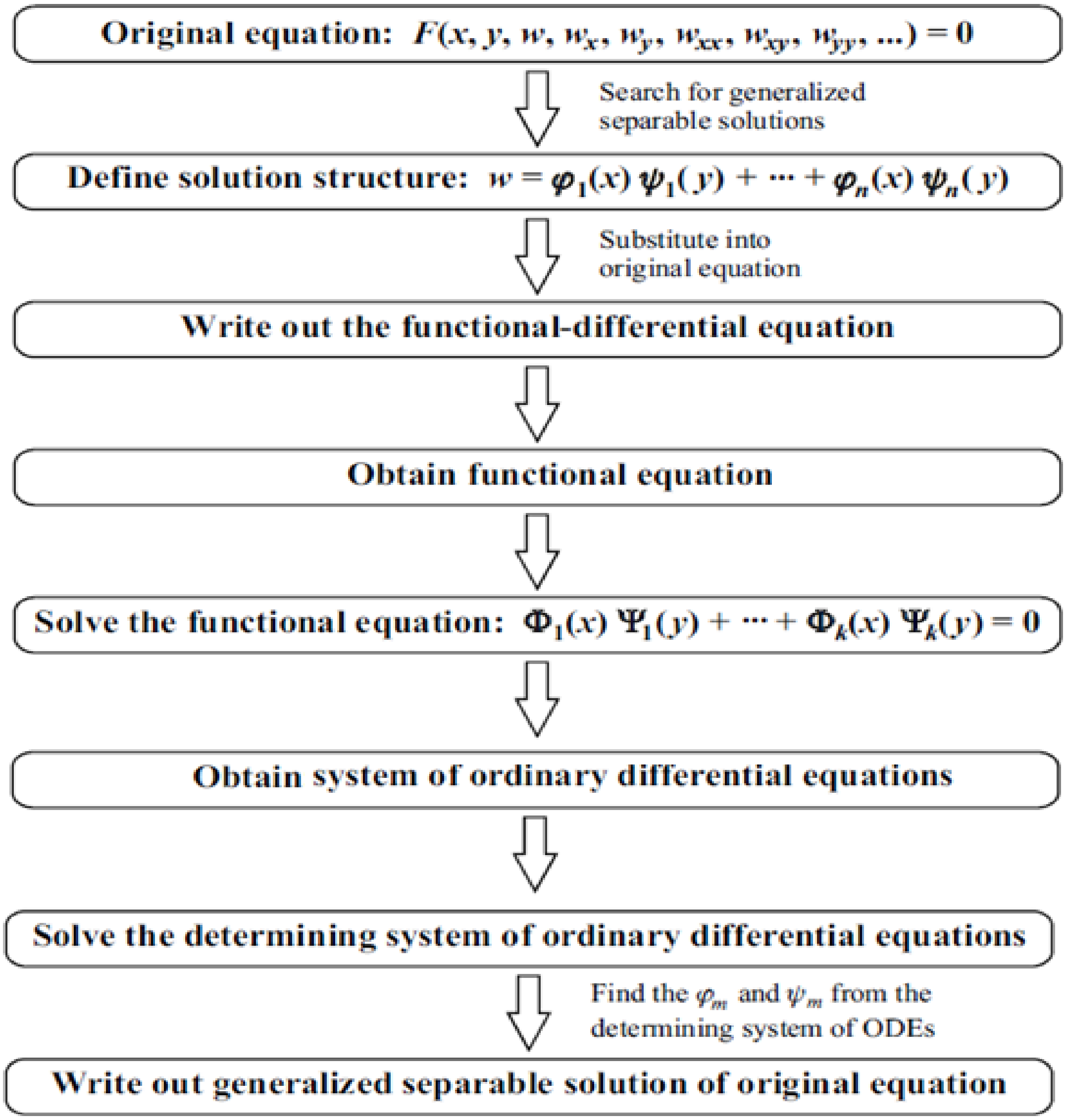

Figure 2.14: General scheme for constructing generalized separable solution

Below we consider specific examples illustrating the applications of the generalized separable method to construct a classical exact solution of nonlinear partial differential equation which will be helpful to understand theory of generalized separable method.

Example: 1

Consider non linear partial differential equation of second order

$$\frac{\partial w}{\partial t} = aw\,\frac{\partial^2 w}{\partial x^2} + f(t)\left(\frac{\partial w}{\partial x}\right)^2 + g(t)\,\frac{\partial w}{\partial x} + h(t)\,w + s(t) \qquad (9)$$

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where $f(t), g(t), h(t)$ and $s(t)$ are arbitrary functions and a is constant.

We look for generalized separable solution of the form

$$w(x,t) = \varphi_1(x)\Psi_1(t) + \varphi_2(x)\Psi_2(t) + \varphi_3(x)\Psi_3(t) \qquad \text{(for n=3)}$$

As discussed in simplified scheme for constructing generalized separable solution, taking

$$\varphi_1(x) = x^2, \qquad \varphi_2(x) = x, \varphi_3(x) = 1$$

Let the solution of w be expressed in generalized separable form quadratic in x as

$$w(x,t) = \Psi_1(t)x^2 + \Psi_2(t)x + \Psi_3(t) \tag{10}$$

Where the functions $\Psi_1(t), \Psi_2(t) and \Psi_3(t)$ are determined by a system of first order ordinary differential equations with variable coefficients obtained by substituting (10) into (9) and equating the coefficients on both the sides as follows:

$$\Psi_1{}'(t) = 2(2f + a)\Psi_1{}^2 + h\Psi_1 \tag{11}$$

$$\Psi_2{}'(t) = (4f\Psi_1 + 2a\Psi_1 + h)\Psi + 2g\Psi_2 \tag{12}$$

$$\Psi_3{}'(t) = (2a\Psi_1 + h)\Psi_3 + f\Psi_2{}^2 + g\Psi_2 + s \tag{13}$$

Equation (11) is a Bernoulli equation so it is easy to integrate. After that, equation (12) and then (13) can be solved with ease, since both are linear in their respective unknowns Ψ_2 and Ψ_3.

Example: 2

Consider Partial differential equation,

$$\frac{\partial w}{\partial t} = aw\frac{\partial^2 w}{\partial x^2} + b\left(\frac{\partial w}{\partial x}\right)^2 + cw^2 + f(t)w + g(t) \tag{14}$$

Let the solution of w be expressed in generalized separable form involving an exponential term of x as

$$w(x,t) = \emptyset(t) + \Psi(t)\exp(\pm\lambda x)\, where \lambda = \left(\frac{-c}{a+b}\right)^{1/2} \tag{15}$$

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where the functions $\emptyset(t)\ and\ \Psi(t)$ are determined by the following system of first order ordinary differential equations with variable coefficients

$$\emptyset'(t) = c\emptyset^2 + f\emptyset + g \quad (16)$$

$$\Psi'(t) = (a\lambda^2\emptyset + 2c\emptyset + f)\Psi \quad (17)$$

Equation (15) is a Riccati equation for $\emptyset = \emptyset(t)$, so it can be reduced to a second order linear equation. The books by Kamke [1977] and Polyanin and Zaitsev[1996] present a large number of solutions to this equation for various *f* and *g*.

In particular, for *g* = 0, equation (15) is a Bernoulli equation, which is easy to integrate.

In another special case, *f, g* = const, a particular solution of (15) is a number$\emptyset = \emptyset_0$, which is a root of the quadratic equation

$$c{\emptyset_0}^2 + f\emptyset_0 + g = 0$$

The substitution$u = \emptyset - \emptyset_0$, leads to a Bernoulli equation.

Given a solution of (15), the solution of equation (16) can be obtained in the form

$$\Psi(t) = C\exp(\int (a\,\lambda^2\emptyset + 2c\emptyset + f)dt) \quad (18)$$

where C is an arbitrary constant.

Substituting the values of $\emptyset(t)\ and\ \Psi(t)$ in (14), we get the required general solution.

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