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Bounds For Weighted Fuzzy Mean Divergence Msn_1 ,Msn_2 and Msn_3

KEYWORDS: Weighted Fuzzy Mean of order t, t ≠ 0, Weighted Fuzzy Arithmetic Mean, Weighted Fuzzy Geometric Mean, Weighted Fuzzy Harmonic Mean, Weighted Fuzzy Root-Square Mean, Weighted Fuzzy N1, N2 and N3 Means. | AMS classification : 2011 94 A 17 : 26 D 15

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ABSTRACT *In the present communication, we will consider bounds for weighted fuzzy mean divergences such as*

(i) $M_{[illegible]}(\mu_A(x_i), \mu_B(x_i) : W) \le 2 M_{[illegible]}(\mu_A(x_i), \mu_B(x_i) : W)$

$\le \frac{3}{2} M_{[illegible]}(\mu_A(x_i), \mu_B(x_i) : W)$ ...(1)

(ii) $M_{[illegible]}(\mu_A(x_i), \mu_B(x_i) : W) \le \frac{4}{3} M_{[illegible]}(\mu_A(x_i), \mu_B(x_i) : W)$

$\le 4 M_{[illegible]}(\mu_A(x_i), \mu_B(x_i) : W)$ ...(2)

and

(iii) $M_{[illegible]}(\mu_A(x_i), \mu_B(x_i) : W) \le \frac{3}{4} M_{[illegible]}(\mu_A(x_i), \mu_B(x_i) : W)$

$\le \frac{3}{2} M_{[illegible]}(\mu_A(x_i), \mu_B(x_i) : W)$ ...(3)

SN1, SN2 and SN3 : | and | and their concavity by exploiting fuzzy weighted f-divergence and calculus. |

1.1 INTRODUCTION :

It is a well known fact that means-Arithmetic, Geometric, Harmonic, Root- Square Mean, Heron's Mean and Logarithmic means have wide applications in Mathematical Seiences, Statistical Analysis, Medical Sciences, Budget Ananlysis, Planning, Environmental Sciences and many others. Recently Kapur and Sharma [12] have applied the Arithmetic, Geometric and Harmonic means in information measures for increasing and decreasing probability distributions and characterized the newly monotonic measures of information ; entropy, inaccuracy and divergence measures and their concavity and convexity.

Recently generalization of these means have created an interest among Liu, H Meng, X-J [15] who introduced contraharmonic mean as Seifferts mean and established different inequalities. Logarithmic mean which can be expressed in terms of Gauss's hypergemetric function $2F_1$, has many applications. For example, a variant of Jonson's functional equation, involving logarthim mean appears in heat conduction problems.

Heronian and Seiffert means have applications in geometry, topology, ordinary differential equations and fuzzy sets. For example, Runge-Kutta methods are based on Heronian mean. A lot of work is being done by Zhi-Hua Zhang and Yu-Dong Wu [22], H.N. Shi, J. Zhang and Da Mao Li [5] and Huan Nan Shi and Zia Zhang [6], Ladislav Mate Jicka [14] have studied many types of means and characterized them and established bounds for them and concavity and convexity of these means.

Taneja [8] considered the differences of means due to the fact that the divergence measures viz. Kullback and Leibler [13] divergence being the difference of inaccuracy measure and Shannon entropy, this divergence has been applied to many areas.

Recently Javier E. Contreras-Reyes and Reinaldo B. Arellano-Valle [10] have applied Kullback-Leibler's [13] divergence measure for multivariate Skew-Normal distributions to study seismic catalogue of the Servicio Sismologico National of Chile containing 6,714 aftershocks on a map [32-40°S] × [69-75.5°E] for a period between 27 February 2010 to 13 July 2011.

Recently Zhi-Hua-Zhang and Yu-Dong Wu [22] have established some new bounds for logarithmic mean viz. :

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$$L(a,b)=\begin{cases}\dfrac{a-b}{\log a-\log b}, & a\neq b\\ a, & a=b\end{cases} \quad (1.1)$$

and the bounds are

$$G\le L\le M_{1/3}\le M_{1/2}\le H_1<M_{2/3}\le A \quad (1.2)$$

stated by J. Ch. Kuang [4]

where

$$\left.\begin{aligned}A=A(a,b)&=\frac{a+b}{2}\\ G=G(a,b)&=\sqrt{ab}\\ H_1=H_1(a,b)&=\frac{a+\sqrt{ab}+b}{3}\end{aligned}\right\} \quad (1.3)$$

and the generalized power type Heronian mean studied by G. Jia and J. D. Cao [4] is :

$$\left.\begin{aligned}H_r(a,b)&=\left[\frac{a^r+(ab)^{\frac{r}{2}}+b^r}{3}\right]^{\frac{1}{r}}, & r\neq 0\\ &=\sqrt{ab}, & r=0\end{aligned}\right\} \quad (1.4)$$

and t-order power mean defined by

$$\left.\begin{aligned}M_t(a,b)&=\left(\frac{a^t+b^t}{3}\right)^{\frac{1}{t}}, & t\neq 0\\ &=\sqrt{ab}, & t=0\end{aligned}\right\} \quad (1.5)$$

The inequality studied by Liu [15] is given by

$G\le L\le M_{t/3}$ (1.6)

Another generalization of Heronian Mean is given by Zh – G.Xiao and Zh – H Zhang [21] as follows :

$$H(a,b;k)=\frac{1}{k+1}\sum_{i=0}^{k}a^{\frac{k-1}{k}}b^{\frac{1}{k}} \quad (1.7)$$

and

$$h(a,b;k)=\frac{1}{k}\sum_{i=1}^{k}a^{\frac{k+1-i}{k+1}}b^{\frac{1}{k+1}} \quad (1.8)$$

As mentioned above, there are many generalizations of means due to their applications

Recently Taneja [8] who generalized mean as mean of order t, $t\neq 0$ as follows :

$$M_t(a,b)=\begin{cases}\left(\dfrac{a^t+b^t}{2}\right)^{1/t}, & t\neq 0\\ \sqrt{ab}, & t=0\\ \max\{ab\}, & t=\infty\\ \min\{ab\}, & t=-\infty\end{cases} \quad (1.9)$$

$\forall$ a.b, t ε $\Re$, a, b > 0 and particular cases

$$\left.\begin{aligned}&t=-1, && M_{-1}(a,b)=H(a,b)=\frac{2ab}{a+b}\\ &t=0, && M_0(a,b)=G(a,b)=\sqrt{ab}\\ &t=\frac{1}{2}, && M_{1/2}(a,b)=N_1(a,b)=\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2\\ &t=1, && M_1(a,b)=A(a,b)=\frac{a+b}{2}\\ &and\\ &t=2, && M_2(a,b)=S(a,b)=\sqrt{\frac{a^2+b^2}{2}}\end{aligned}\right\} \quad (1.10)$$

Also defined some mixed means such as :

$$N_1(a,b)=\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \quad (1.11)$$

$$N_2(a,b)=\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right)$$

$$=N_1(a,b)A(a,b) \quad (1.12)$$

$$N_3(a,b)=\left(\frac{a+\sqrt{ab}+b}{3}\right)$$

$$=\frac{2A(a,b)+G(a,b)}{3} \quad (1.13)$$

$= H_1$ (J. Ch. Kaung)

and

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$$S(a,b)=\sqrt{\frac{a^2+b^2}{2}} \qquad (1.14)$$

So, Taneja [8] considered the difference of means exploiting Csiszar's [7] f-divergence for probabilistic divergences, and has studied convexities of the difference- divergence measures of means stated above.

The motivation for this study, is due to the fact that Singh and Tomar [8] have already studied fuzzy means of different kinds. Here in this communication our objective is to enhance the studies further considering mixed fuzzy means for weighted distributions, their difference-divergences to establish new inequalities and bounds among them and their concavity, particularly in SN_1, SN_2 and SN_3, which are defined in the next section along with others needed.

Importance of the event or experiment has been the outlook of every human being, therefore, we utilize, the weighted distribution corresponding to fuzzy set theoretic distribution and consider the following fuzzy information scheme :

$$FS_1=\begin{bmatrix} E_1 & E_2 \ldots\ldots\ldots\ldots E_n \\ \mu_A(E_1) & \mu_A(E_2)\ldots\ldots\ldots\mu_A(E_n) \\ w_1 & w_2\ldots\ldots\ldots\ldots\ldots w_n \end{bmatrix} \qquad (1.15)$$

Hence the weighted fuzzy entropy is given by

$$F(\mu_A(E_i);W)=-\sum w_i\left[\mu_A(E_i)\log\mu_A(E_i)\right.$$

$$\left.+(1-\mu_A(E_i)\log(1-\mu_A(E_i))\right] \qquad (1.16)$$

Since the basic objective is to study divergence of the fuzzy means weighted, so we consider the revised fuzzy information scheme as :

$$F_2S_2=\begin{bmatrix} E_1 & E_2 \ldots\ldots\ldots\ldots E_n \\ \mu_A(E_1) & \mu_A(E_2)\ldots\ldots\ldots\mu_A(E_n) \\ \mu_B(E_1) & \mu_B(E_2)\ldots\ldots\ldots\mu_B(E_n) \\ w_1 & w_2\ldots\ldots\ldots\ldots\ldots w_n \end{bmatrix} \qquad (1.17)$$

and define the weighted fuzzy divergence as :

$$F_D(A\|B;W)=\sum_{i=1}^{n} w_i\left[\mu_A(E_i)\log\frac{\mu_A(E_i)}{\mu_B(E_i)}\right.$$

$$\left.+(1-\mu_A(E_i))\frac{(1-\mu_A(E_i))}{(1-\mu_B(E_i))}\right] \qquad (1.18)$$

A lot of literature is available for fuzzy entropy, information, divergence, inaccuracy and other fuzzy information indices in De Luca and Termini , Bezdek, Ebanks, Pal & Pal, Pal and Bezdek, Bhandari and Pal, Kapur, Hooda, Omparkash et. al. Gurdial et. al. Loo Kauf Mann, Yagger, Kosko, Parade and many others.

For more detail c. f Nikhil R. Pal and James C. Bezdek [16] for different types of fuzzy entropies, parametric and non-parametric. For divergence measures, c.f Kapur [11], Parkash [17], Singh and Tomar [20] and Bhatia and Singh [2]. Recently Priti, Sharma and Singh [18] considered fuzzy weighted divergence-differences and studied their concavity. Inequalities among weighted fuzzy means have been studied by authors [18].

Since in this communication, our objective is to consider divergence of fuzzy means as difference of means, so we define the weighted t-order fuzzy mean as follows :

$$M_t(\mu_A;\mu_B;W)=\sum_{i=1}^{n} w_i\left[\left\{\frac{\mu_A^t(E_i)+\mu_B^t(E_i)}{2}\right\}^{\frac{1}{t}}\right.$$

$$\left.+\left\{\frac{(1-\mu_A(E_i))^t+(1-\mu_B(E_i))^t}{2}\right\}^{\frac{1}{t}}\right],\quad t\neq 0 \qquad (1.19)$$

and the particular cases for different values of t, corresponding to equation (1.9) and (1.10) as follows:

Particular Cases

Sr.No.	t	Mean	Fuzzy Expression
1.	When $t = -1$	Fuzzy Harmonic Mean $=M_{-1}(\mu_A, \mu_B; W)$ $=H(\mu_A(E_i), \mu_B(E_i); W)$	$=\sum_{i=1}^{n} w_i\left\{\frac{2\mu_A(E_i)+\mu_B(E_i)}{\mu_A(E_i)+\mu_B(E_i)}+\frac{2(1-\mu_A(E_i))(1-\mu_B(E_i))}{2-\mu_A(E_i)-\mu_B(E_i)}\right\}$
2.	When $t = 0$	Fuzzy Geometric Mean $= M_0(\mu_A, \mu_B;W)$ $=G(\mu_A(E_i), \mu_B(E_i); W)$	$=\sum_{i=1}^{n} w_i\sqrt{\mu_A(E_i)\mu_B(E_i)+((1-\mu_A(E_i)(1-\mu_B(E_i))}$
3.	When $t = 1/2$	Fuzzy $M_{1/2}(\mu_A, \mu_B;W)$ $=N_1(\mu_A(E_i), \mu_B(E_i); W)$	$=\sum_{i=1}^{n} w_i\left\{\left(\frac{\sqrt{\mu_A(E_i)}+\sqrt{\mu_B(E_i)}}{2}\right)^2 \left(\frac{\sqrt{(1-\mu_A(E_i))}+\sqrt{(1-\mu_B(E_i))}}{2}\right)^2\right\}$
4.	When $t = 1$	Fuzzy Arithmetic Mean $= M_1(\mu_A, \mu_B; W)$ $=A(\mu_A(E_i), \mu_B(E_i); W)$	$=\sum_{i=1}^{n} w_i\left\{\frac{\mu_A(E_i)+\mu_B(E_i)}{2}+\frac{2-\mu_A(E_i)-\mu_B(E_i)}{2}\right\}$
5.	When $t = 2$	Fuzzy Root-Square Mean$= M_2(\mu_A,\mu_B;W)$ $= S(\mu_A(E_i), \mu_B(E_i); W)$	$=\sum_{i=1}^{n} w_i\left\{\frac{\sqrt{\mu_A^2(E_i)+\mu_A^2(E_i)}}{2} \sqrt{\frac{(1-\mu_A(E_i))^2+(1-\mu_B(E_i))^2}{2}}\right\}$
6.	When $t=-\infty$	Fuzzy $M_{-\infty}(\mu_A, \mu_B; W)$	$= \min(\mu_A(E_i), \mu_B(E_i); W)$
7.	When $t = \infty$	Fuzzy $M_{\infty}(\mu_A, \mu_B; W)$	$= \max(\mu_A(E_i), \mu_B(E_i); W)$

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MIXED WEIGHTED FUZZY MEAN MEASURES

1. $N_2(\mu_A(E_i),\mu_B(E_i);W)$

$$=\sqrt{N_1(\mu_A(E_i),\mu_B(E_i);W)A(\mu_A(E_i),\mu_B(E_i);W)}$$

$$=\sum_{i=1}^{n} w_i\left[\left\{\left(\frac{\sqrt{\mu_A(E_i)}+\sqrt{\mu_B(E_i)}}{2}\right)^2+\left(\frac{\sqrt{(1-\mu_A(E_i))}+\sqrt{(1-\mu_B(E_i))}}{2}\right)^2\right\}\left\{\frac{\mu_A(E_i)+\mu_B(E_i)}{2}+\frac{2-\mu_A(E_i)-\mu_B(E_i)}{2}\right\}\right]$$

or

$$=\sum_{i=1}^{n} w_i\left[\left\{\frac{\sqrt{\mu_A(E_i)}+\sqrt{\mu_B(E_i)}}{2}\right\}\left\{\frac{\sqrt{\mu_A(E_i)+\mu_B(E_i)}}{2}\right\}+\left\{\frac{\sqrt{(1-\mu_A(E_i))}+\sqrt{(1-\mu_B(E_i))}}{2}\right\}\left\{\frac{\sqrt{2-\mu_A(E_i)-\mu_B(E_i)}}{2}\right\}\right] \quad (1.20)$$

2. $N_3(\mu_A(E_i),\mu_B(E_i);W)$

$$=\frac{2A(\mu_A(E_i),\mu_B(E_i);W)+G(\mu_A(E_i),\mu_B(E_i);W)}{3}$$

$$=\sum_{i=1}^{n} w_i\left[\frac{\{\mu_A(E_i)+\mu_B(E_i)+(1-\mu_A(E_i))(1-\mu_B(E_i))\}}{3}+\left\{\frac{\sqrt{\mu_A(E_i)+\mu_B(E_i)+(1-\mu_A(E_i))(1-\mu_B(E_i))}}{3}\right\}\right]$$

or

$$=\sum_{i=1}^{n} w_i\left[\left\{\frac{\mu_A(E_i)+\mu_B(E_i)+\sqrt{\mu_A(E_i)+\mu_B(E_i)}}{3}\right\}+\left\{\frac{2-\mu_A(E_i)-\mu_B(E_i)+\sqrt{(1-\mu_A(E_i))(1-\mu_B(E_i))}}{3}\right\}\right] \quad (1.21)$$

In the next section, we present the weighted fuzzy mean-divergence measures.

SECTION – 2

2.1 Basic Mean Difference Divergence Measures

In fact, the mean-difference-divergences to be utilized for the bounds to be discussed in the present papers are as follows :

Sr. No.	Measure	Expression
1.	$M_{SA}(\mu_A(E_i), \mu_B(E_i);W)$ $=M_{SA}(A \parallel B; W)$	$=\sum_{i=1}^{n} w_i \left[\sqrt{\frac{\mu_A^2(E_i)+\mu_B^2(E_i)}{2}}+\sqrt{\frac{(1-\mu_A(E_i))^2+(1-\mu_B(E_i))^2}{2}} -\left(\frac{\mu_A(E_i)+\mu_B(E_i)}{2}\right)-\left(\frac{2-\mu_A(E_i)-\mu_B(E_i)}{2}\right)\right]$
2.	$M_{SN_2}(\mu_A(E_i),\mu_B(E_i);W)$ $=M_{SN_2}(A\parallel B;W)$	$=\sum_{i=1}^{n} w_i \left[\sqrt{\frac{\mu_A^2(E_i)+\mu_B^2(E_i)}{2}}+\sqrt{\frac{(1-\mu_A(E_i))^2+(1-\mu_B(E_i))^2}{2}} -\left(\frac{\sqrt{\mu_A(E_i)}+\sqrt{\mu_B(E_i)}}{2}\right)\sqrt{\frac{\mu_A(E_i)+\mu_B(E_i)}{2}} -\left(\frac{\sqrt{(1-\mu_A(E_i))}}{2}+\frac{\sqrt{(1-\mu_B(E_i))}}{2}\right)\sqrt{\frac{2-\mu_A(E_i)-\mu_B(E_i)}{2}}\right]$
3.	$M_{AN_2}(\mu_A(E_i),\mu_B(E_i);W)$ $=M_{AN_2}(A\parallel B;W)$	$=\sum_{i=1}^{n} w_i \left[\frac{\mu_A(E_i)+\mu_B(E_i)}{2}+\left(\frac{2-\mu_A(E_i)-\mu_B(E_i)}{2}\right) -\left(\frac{\sqrt{\mu_A(E_i)}+\sqrt{\mu_B(E_i)}}{2}\right)\times\left(\sqrt{\frac{\mu_A(E_i)+\mu_B(E_i)}{2}}\right) -\left(\frac{\sqrt{1-\mu_A(E_i)}+\sqrt{1-\mu_B(E_i)}}{2}\right)\left(\sqrt{\frac{2-\mu_A(E_i)-\mu_B(E_i)}{2}}\right)\right]$
4.	$M_{SH}(\mu_A(E_i),\mu_B(E_i);W)$ $=M_{SH}(A\parallel B;W)$	$=\sum_{i=1}^{n} w_i \left[\sqrt{\frac{\mu_A^2(E_i)+\mu_B^2(E_i)}{2}}+\sqrt{\frac{(1-\mu_A(E_i))^2+(1-\mu_B(E_i))^2}{2}} -\frac{2\mu_A(E_i)\mu_B(E_i)}{\mu_A(E_i)+\mu_B(E_i)}-\frac{2(1-\mu_A(E_i))(1-\mu_B(E_i))}{2-\mu_A(E_i)-\mu_B(E_i)}\right]$
5.	$M_{SN_1}(\mu_A(E_i),\mu_B(E_i);W)$ $=M_{SN_1}(A\parallel B;W)$	$=\sum_{i=1}^{n} w_i \left[\sqrt{\frac{\mu_A^2(E_i)+\mu_B^2(E_i)}{2}}+\sqrt{\frac{(1-\mu_A(E_i))^2+(1-\mu_B(E_i))^2}{2}} -\left(\frac{\sqrt{\mu_A(E_i)}+\sqrt{\mu_B(E_i)}}{2}\right)^2-\left(\frac{\sqrt{1-\mu_A(E_i)}+\sqrt{1-\mu_B(E_i)}}{2}\right)^2\right]$
6.	$M_{SG}(\mu_A(E_i),\mu_B(E_i);W)$ $=M_{SG}(A\parallel B;W)$	$=\sum_{i=1}^{n} w_i \left[\sqrt{\frac{\mu_A^2(E_i)+\mu_B^2(E_i)}{2}}+\sqrt{\frac{(1-\mu_A(E_i))^2+(1-\mu_B(E_i))^2}{2}} -\sqrt{\mu_A(E_i)\mu_B(E_i)}-\sqrt{(1-\mu_A(E_i))(1-\mu_B(E_i))}\right]$
7.	$M_{SH_3}(\mu_A(E_i),\mu_B(E_i);W)$ $=M_{SN_3}(A\parallel B;W)$	$=\sum_{i=1}^{n} w_i \left[\sqrt{\frac{\mu_A^2(E_i)+\mu_B^2(E_i)}{2}}+\sqrt{\frac{(1-\mu_A(E_i))^2+(1-\mu_B(E_i))^2}{2}}\right.$

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		$-\frac{\mu_A(E_i)+\mu_B(E_i)+\sqrt{\mu_A(E_i)\mu_B(E_i)}}{3}$ $-\frac{2-\mu_A(E_i)-\mu_B(E_i)+\sqrt{(1-\mu_A(E_i))(1-\mu_B(E_i))}}{3}\Bigg]$

SECTION – 3

Functional Forms, First and Second Derivatives

Setting $\mu_A(E_i)=$ x, $\mu_B(E_i)=$ ½ in the above table, we have

1. $M_{SA}(A\|B;W)=wf_{SA}(x)$, where

$$f_{SA}(x)=\left[\sqrt{\frac{x^2+\frac{1}{4}}{2}}+\sqrt{\frac{(1-x)^2+\frac{1}{4}}{2}}-1\right] \quad (3.1)$$

$$f'_{SA}(x)=\frac{x}{2}\left[\frac{x^2+\frac{1}{4}}{2}\right]^{-\frac{1}{2}}-\frac{(1-x)}{2}\left(\frac{(1-x)^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}} \quad (3.2)$$

$$f''_{SA}(x)=\left[\left[\frac{1}{2}\left[\frac{x^2+\frac{1}{4}}{2}\right]\right]^{\frac{1}{2}}+\frac{x^2}{4}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{3}{2}}\right.$$

$$\left.+\frac{1}{2}\left(\frac{(1-x)^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}-\frac{(1-x)^2}{4}\left(\frac{(1-x)^2+\frac{1}{4}}{2}\right)^{-\frac{3}{2}}\right] \quad (3.3)$$

2. $M_{SN_1}(A\|B;W)=wf_{SN_1}(x)$, where

$$f_{SN_1}(x)=\left[\sqrt{\frac{x^2+\frac{1}{4}}{2}}+\sqrt{\frac{(1-x)^2+\frac{1}{4}}{2}}\right.$$

$$\left.-\frac{x+\frac{1}{2}\sqrt{\frac{x}{2}}}{3}-\frac{\frac{3}{2}-x+\sqrt{\frac{1-x}{2}}}{3}\right] \quad (3.4)$$

$$f'_{SN_1}(x)=\left[\frac{x}{2}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}-\frac{(1-x)}{2}\left(\frac{(1-x)^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}\right.$$

$$-\frac{1}{12}\left(\frac{x}{2}\right)^{-\frac{1}{2}}+\frac{1}{12}\left(\frac{1-x}{2}\right)^{-\frac{1}{2}} \quad (3.5)$$

$$f''_{SN_1}(x)=\left[P+\frac{1}{48}\left(\frac{x}{2}\right)^{-\frac{3}{2}}+\frac{1}{48}\left(\frac{1-x}{2}\right)^{-\frac{3}{2}}\right] \quad (3.6)$$

where $P=\frac{1}{2}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}+\frac{1}{2}\left(\frac{(1-x)^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}$

$$-\frac{x^2}{4}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{3}{2}}-\frac{(1-x)^2}{4}\left(\frac{(1-x)^2+\frac{1}{4}}{2}\right)^{-\frac{3}{2}} \quad (3.7)$$

3. $M_{SN_2}(A\|B;W)=wf_{SN_2}(x)$, where

$$f_{SN_2}(x)=\left[\sqrt{\frac{x^2+\frac{1}{4}}{2}}+\sqrt{\frac{(1-x)^2+\frac{1}{4}}{2}}-\left(\frac{\sqrt{x}+\sqrt{\frac{1}{2}}}{2}\right)\right.$$

$$\left.\left(\sqrt{\frac{x+\frac{1}{2}}{2}}\right)-\frac{\sqrt{(1-x)}}{2}+\frac{\sqrt{\frac{1}{2}}\sqrt{\frac{3}{-2}-x}}{2}\right] \quad (3.8)$$

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$$f'_{SN_2}(x)=\frac{x}{2}\left[\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}-\frac{(1-x)}{2}\left(\frac{(1-x)^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}\right.$$

$$-\frac{1}{4\sqrt{x}}\sqrt{\frac{x+\frac{1}{2}}{4}}-\frac{1}{4}\left(\frac{\sqrt{x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\frac{x+\frac{1}{2}}{2}\right)^{-\frac{1}{2}}$$

$$\left.+\frac{1}{4\sqrt{1-x}}\sqrt{\frac{\frac{3}{2}-x}{4}}+\frac{1}{4}\left(\frac{\sqrt{1-x}+\frac{1}{\sqrt{2}}}{2}\right)\left(\frac{\frac{3}{2}-x}{2}\right)^{-\frac{1}{2}}\right] \quad (3.9)$$

$$f''_{SN_3}(x)=P+Q \quad (3.10)$$

where

$$Q=\frac{x\sqrt{\frac{x}{2}}+\frac{1}{4}}{8\sqrt{2}x^{3/2}\left(x+\frac{1}{2}\right)^{3/2}}+\frac{(1-x)\sqrt{\frac{1-x}{2}}+\frac{1}{4}}{8\sqrt{2}(1-x)^{3/2}\left(\frac{3}{2}-x\right)^{3/2}}$$

..(3.11)

and P is given by (3.7).

4. $M_{SN_3}(A\,||\,B;W)=wf_{SN_3}(x)$, where

$$f_{SN_3}(x)=\left[\sqrt{\frac{x^2+\frac{1}{4}}{2}}+\sqrt{\frac{(1-x)^2+\frac{1}{4}}{2}}\right.$$

$$\left.-\frac{\left(x+\frac{1}{2}\right)+\sqrt{\frac{x}{2}}}{3}\,\frac{\frac{3}{2}-x+\sqrt{\frac{1-x}{2}}}{3}\right] \quad (3.12)$$

$$f'_{SN_3}(x)=\left[\frac{x}{2}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}-\frac{1-x}{2}\left(\frac{(1-x)^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}\right.$$

$$\left.-\frac{1}{12}\left(\frac{x}{2}\right)^{-\frac{1}{2}}+\frac{1}{12}\left(\frac{1-x}{2}\right)^{-\frac{1}{2}}\right] \quad (3.13)$$

$$f''_{SN_3}(x)=\left[P+\frac{1}{48}\left(\frac{x}{2}\right)^{-\frac{3}{2}}+\frac{1}{48}\left(\frac{1-x}{2}\right)^{-\frac{3}{2}}\right] \quad (3.14)$$

5. $M_{SH}(A\,||\,B;W)=w\,f_{SH}(x)$, where

$$f_{SH}(x)=\left[\sqrt{\frac{x^2+\frac{1}{4}}{2}}+\frac{\sqrt{(1-x)^2+\frac{1}{4}}}{2}-\frac{x}{x+\frac{1}{2}}-\frac{1-x}{\frac{3}{2}-x}\right] \quad (3.15)$$

$$f'_{SH}(x)=\left[\frac{x}{2}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}-\left(\frac{1-x}{2}\right)\left(\frac{1-x}{2}\right)\right.$$

$$\left.\left(\frac{(1-x)^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}-\frac{1}{2\left(x+\frac{1}{2}\right)^2}+\frac{1}{2\left(\frac{3}{2}-x\right)^2}\right] \quad (3.16)$$

$$f''_{SH}(x)=\left[P+\frac{1}{\left(x+\frac{1}{2}\right)^3}+\frac{1}{\left(\frac{3}{2}-x\right)^3}\right] \quad (3.17)$$

6. $M_{SG}(A\,||\,B;W)=w\,f_{SG}(x)$, where

$$f_{SG}(x)=\left[\sqrt{\frac{x^2+\frac{1}{4}}{2}}+\sqrt{\frac{(1-x)^2+\frac{1}{4}}{2}}-\sqrt{\frac{x}{2}}-\sqrt{\frac{1-x}{2}}\right] \quad (3.18)$$

$$f'_{SG}(x)=\left[\frac{x}{2}\left(\frac{x^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}-\left(\frac{1-x}{2}\right)\left(\frac{(1-x)^2+\frac{1}{4}}{2}\right)^{-\frac{1}{2}}\right.$$

$$\left.-\frac{1}{4}\left(\frac{x}{2}\right)^{-\frac{1}{2}}+\frac{1}{4}\left(\frac{1-x}{2}\right)^{-\frac{1}{2}}\right] \quad (3.19)$$

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$$f''_{SG}(x)=\left[P+\frac{1}{16}\left(\frac{x}{2}\right)^{-\frac{3}{2}}+\frac{1}{16}\left(\frac{1-x}{2}\right)^{-\frac{3}{2}}\right] \quad (3.20)$$

where P is given by (3.7)

7. $M_{AN_2}(A\|B;W)=wf_{AN_2}(x)$, where

$$f_{AN_2}(x)=\left[1-\left(\frac{\sqrt{x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\frac{\sqrt{x+\frac{1}{2}}}{2}\right)-\left(\frac{\sqrt{1-x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\frac{\sqrt{\frac{3}{2}-x}}{2}\right)\right] \quad (3.21)$$

$$f'_{AN_2}(x)=\left[\frac{1}{4}\sqrt{x}\left(\frac{\sqrt{x+\frac{1}{2}}}{2}\right)+\frac{1}{4}\left(\frac{\sqrt{x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\frac{x+\frac{1}{2}}{2}\right)^{-\frac{1}{2}}+\frac{1}{4\sqrt{1-x}}\left(\frac{\sqrt{\frac{3}{2}-x}}{2}\right)+\frac{1}{4}\left(\frac{\sqrt{1-x}+\sqrt{\frac{1}{2}}}{2}\right)\left(\frac{\frac{3}{2}-x}{2}\right)^{-\frac{1}{2}}\right] \quad (3.22)$$

$$f''_{AN_2}(x)=[Q] \quad (3.23)$$

where Q is given by (3.11)

SECTION-4

Csiszar's f-Divergence Extended For Fuzzy f-Divergence

Csiszar's f-Divergence

Definition : If the function $f:(0,\infty)\rightarrow R$ is convex and normalized i.e. $f(1)=0$, then the f-divergence is given by:

$$C_f(P\|Q)=\sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right) \quad (4.1)$$

Extension for Weighted f-Divergence

Definition : Let f-divergence defined by Csiszar[7] be

$$C_f(P\|Q:1)=\sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right)$$

satisfied by $f:(0,\infty)\rightarrow R$, then the weighted f-divergence is defined as :

$$C_f(P\|Q;W)=\sum_{i=1}^{n} w_i q_i f\left(\frac{p_i}{q_i}\right) \quad (4.2)$$

where $W=(w_1, w_2, \ldots, w_n)$, $w_i>0$ $\forall\, i=1,2,\ldots,n$

corresponding to the probability distributions

$P=(p_1, p_2, \ldots, p_n)$

$Q=(q_1, q_2, \ldots, q_n)$

where $0\le p_i\le 1$, $0\le q_i\le 1$.

$$\sum_{i=1}^{n} p_i=\sum_{i=1}^{n} q_i=1$$

Extension for Fuzzy f-Divergence

Analogous to Csiszar's probabilistic f-divergence, we define fuzzy f-divergence as below :

Fuzzy f-Divergence : Let $f:(0,\infty)\rightarrow R$ be a convex and normalized function such that

$$C_f(A\|B)=\sum_{i=1}^{n}\mu_B(x_i) f\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right) \quad (4.3)$$

Weighted Fuzzy f-Divergence : Let

$$F_f(A\|B)=\sum_{i=1}^{n}\mu_B(x_i) f\left(\frac{\mu_A}{\mu_B}\right)$$

be the Fuzzy f-divergence, then the **Weighted Fuzzy f-divergence** is defined as

$$F_f(A\|B;W)=\sum_{i=1}^{n}w_i\mu_B(x_i)f\begin{pmatrix}\mu_A(x_i)\\ \mu_B(x_i)\end{pmatrix} \quad (4.4)$$

Taneja[8] applied the following results :

Lemma : Let $f_1, f_2 : I \subseteq \Re_+ \to \Re$ be two convex functions that are normalized i.e. $f_1(1) = f_2(1) = 0$ and suppose the assumptions :

(i) f_1 and f_2 are twice differentiable on [a, b].

(ii) There exist the real constants m and M such that m < M and

$$m\le\frac{f_1''(x)}{f_2''(x)}\le M,\ f_2''(x)>0,\forall\ x\in(a,b) \quad (4.5)$$

then

$$mC_{f_2}(P\|Q)\le C_{f_1}(P\|Q)\le MC_{f_2}(P\|Q) \quad (4.6)$$

Considering (4.3), the Fuzzy Extension for (4.6) is :

$$mF_{f_2}(A\|B)\le F_{f_1}(A\|B)\le MF_{f_2}(A\|B) \quad (4.7)$$

and for Weighted Fuzzy f-Divergence, we have

$$mF_{f_2}(A\|B;W)\le F_{f_1}(A\|B;W)\le MF_{f_2}(A\|B;W) \quad (4.8)$$

The above mentioned results (4.4), (4.5) and (4.8), have been exploited in section 5 for bounds, inequalities and concavity.

SECTION – 5

5.1. Concavity and Bounds for M_{SN_1}, M_{SN_2} and M_{SN_3}

In this section we have considered the following inequalities/bounds:

(i) $M_{SH}(\mu_A,\mu_B;W)\le 2M_{SN_1}(\mu_A,\mu_B;W)$

$$\le\frac{3}{2}M_{SG}(\mu_A,\mu_B;W) \quad (5.1)$$

(ii) $M_{SA}(\mu_A,\mu_B;W)\le\frac{4}{5}M_{SN_2}(\mu_A,\mu_B;W)$

$$\le 4M_{AN_2}(\mu_A,\mu_B;W) \quad (5.2)$$

and

(iii) $M_{SA}(\mu_A,\mu_B;W)\le\frac{3}{4}M_{SN_3}(\mu_A,\mu_B;W)$

$$\le\frac{2}{3}M_{SN_1}(\mu_A,\mu_B;W) \quad (5.3)$$

which have been established through prepositions as well as their concavity property which is needed in case of detailed analysis. Now we consider the inequality (5.1) through the following prepositions.

Preposition 1 : The following inequality holds good. i.e. Lower Bound for M_{SN_1}

i.e.

$$M_{SH}(\mu_A(x_i),\mu_B(x_i);W)\le 2M_{SN_1}(\mu_A(x_i),\mu_B(x_i);W)$$

or

$$M_{SH}(A\|B;W)\le 2M_{SN_1}(A\|B;W) \quad (5.4)$$

Proof : Let us define

$$h_{SH-SN_1}(x)=\frac{f_{SH}''(x)}{f_{SN_1}''(x)},\qquad \forall x\in(0,1)$$

$$=\frac{P+\dfrac{1}{\left(x+\frac{1}{2}\right)^3}+\dfrac{1}{\left(\frac{3}{2}-x\right)^3}}{P+\frac{1}{4}(2x)^{-\frac{3}{2}}+\frac{1}{4}\left[2(1-x)\right]^{-\frac{3}{2}}} \quad (5.5)$$

Differentiating (5.5) w.r.t. x, we get

$$h'_{SH-SN_1}(x)=\frac{D\left[P'-\dfrac{3}{\left(x+\frac{1}{2}\right)^4}+\dfrac{3}{\left(\frac{3}{2}-x\right)^4}\right]}{D^2}$$

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$$\frac{-N\left[P'-\frac{3}{4}(2x)^{-\frac{3}{2}}+\frac{3}{4}\left(2(1-x)\right)^{-\frac{5}{2}}\right]}{D^2} \quad (5.6)$$

where N and D stands for numerator and denominator of (5.5).

From (5.6), we observe that

$$h'_{SH-SN_1}(x)\begin{cases}\geq 0, & when\ x\leq\frac{1}{2}\\ \leq 0, & when\ x\geq\frac{1}{2}\\ M=\underset{x\in(0,1)}{Sup}\,h_{SH-SN_1}(x) \\ =h_{SH-SN_1}\left(\frac{1}{2}\right)=2\end{cases} \quad (5.7)$$

Hence from (5.6) & (5.7), we observe that the function $h_{SH-SN_1}(x)$ is increasing in x ε (0, ½) and decreasing in x ε (½,1). Hence CONCAVE in (0,1), ∀ x ε (0,1), w > 0.

Now applying (4.8) for M_{SH}(A||B; W) and M_{SN_1} (A||B;W), together with (5.7), we get the required inequality i.e.

$$M_{SH}(A \,||\, B;\, W) \leq 2\, M_{SN_1}(A \,||\, B;\, W).$$

Preposition 2 : The following inequality holds good :

$$M_{SN_1}(\mu_A(x_i),\mu_B(x_i);W)\leq\frac{3}{4}M_{SG}(\mu_A(x_i),\mu_B(x_i);W)$$

or

$$M_{SN_1}(A\,||\,B;W)\leq\frac{3}{4}M_{SG}(A\,||\,B;W) \quad (5.8)$$

Proof : Let us define

$$h_{SN_1-SG}(x)=\frac{f''_{SN_1}(x)}{f''_{SG}(x)}, \qquad \forall x\in(0,1)$$

$$=\frac{P+\frac{1}{4}(2x)^{-\frac{3}{2}}+\frac{1}{4}(2(1-x))^{-\frac{3}{2}}}{P+\frac{1}{16}\left(\frac{x}{2}\right)^{-\frac{3}{2}}+\frac{1}{16}\left(\frac{1-x}{2}\right)^{-\frac{3}{2}}} \quad (5.9)$$

Differentiating (5.9) w.r.t., we get

$$h'_{SN_1-SG}(x)=\frac{D\left[P'-\frac{3}{4}(2x)^{-\frac{5}{2}}+\frac{3}{4}\left(2(1-x)\right)^{-\frac{5}{2}}\right]}{D^2}$$

$$-\frac{N\left[P'-\frac{3}{64}\left(\frac{x}{2}\right)^{-\frac{5}{2}}+\frac{3}{4}\left(\frac{1-x}{2}\right)^{-\frac{5}{2}}\right]}{D^2} \quad (5.10)$$

From (5.10), we observe that

$$h'_{SN_1-SG}(x)\begin{cases}\geq 0, & when\ x\leq\frac{1}{2}\\ \leq 0, & when\ x\geq\frac{1}{2}\\ M=\underset{x\in(0,1)}{Sup}\,h_{SN_1-SG}(x) \\ =h_{SN_1-SG}\left(\frac{1}{2}\right) \\ =\frac{3}{4}\end{cases} \quad (5.11)$$

From (5.10) and (5.11), we observe that h_{SN_1-SG} (x) is increasing in x ε (0, ½) and decreasing in x ε (½, 1). Hence CONCAVE in x ε (0, 1), w > 0.

Now Applying (4.8) for M_{SN_1} (A||B;W) and M_{SG}(A||B; W) together with (5.11), we get the required inequality i.e.

$$M_{SN_1}(A\,||\,B;W)\leq\frac{3}{4}M_{SG}(A\,||\,B;W).$$

Now combing proposition (1) and Proposition (2), we get the required inequality :

$$M_{SH}(A\,||\,B;W)\leq 2M_{SN_1}(A\,||\,B;W)\leq\frac{3}{2}M_{SG}(A\,||\,B;W)$$

Preposition 3 : The following inequality holds good, i.e. Lower Bound for M_{SN_2} :

$$M_{SA}(\mu_A,\mu_B;W) \le \frac{4}{5} M_{SN_2}(\mu_A,\mu_B;W) \qquad (5.12)$$

Proof : We define

$$h_{SA-SN_2}(x) = \frac{f''_{SA}(x)}{f''_{SN_2}(x)}, \qquad \forall x \in (0,1)$$

$$= \frac{P}{P+Q} \qquad (5.13)$$

Differentiating (5.13) w.r.t. x, we get

$$h'_{SA-SN_2}(x) = \frac{(P+Q)P' - P(P+Q)'}{(P+Q)^2} \qquad (5.14)$$

Setting the values of P, Q, P′, Q′, we conclude that

$$h'_{SA-SN_2}(x) = \begin{cases} \ge 0, when \ x \le \frac{1}{2} \\ \le 0, when \ x \ge \frac{1}{2} \\ M = \sup_{x\in(0,1)} h_{SA-SN_2}(x) \\ = h_{SA-SN_2}\left(\frac{1}{2}\right) \\ = \frac{4}{5} \end{cases} \qquad (5.15)$$

We observe from (5.14) and (5.15) that the function h_{SA-SN_2}(x) is increasing in x ∈ (0, ½) and decreasing in x ∈ (½ , 1), hence CONCAVE in x ∈ (0, 1), w > 0.

Applying (4.8)

$$mF_{f_2}(A\|B;W) \le F_{f_1}(A\|B;W) \le MF_{f_2}(A\|B;W) \qquad (5.16)$$

due to Singh and Tomar [20], being an extension of Csiszar's [7] and Taneja [8] for M_{SA}(A || B; W) and M_{SN_2} (A || B; W) alongwith (5.15), we get required inequality (5.12).

Preposition 4 : The following inequality holds good : Upper Bound for M_{SN_2}

$$\frac{4}{5} M_{SN_2}(A\|B;W) \le M_{AN_2}(A\|B;W) \qquad (5.17)$$

Proof : We define

$$h_{SN_2-AN_2}(x) = \frac{f''_{SN_2}(x)}{f''_{AN_2}(x)}, \qquad \forall x \in (0,1)$$

$$= \frac{P+Q}{Q} \qquad (5.18)$$

where P and Q are given by (3.7) and (3.11).

Differentiating (5.18) w.r.t. ℓ, we get

$$h'_{SN_2-AN_2}(x) = \frac{Q(P+Q)' - Q'(P+Q)}{Q^2} \qquad (5.19)$$

We observe from (5.19)

$$h'_{SN_2-AN_2}(x) = \begin{cases} \ge 0, when \ x \le \frac{1}{2} \\ \le 0, when \ x \ge \frac{1}{2} \\ M = \sup_{x\in(0,1)} h_{SN_2-AN_2}(x) \\ = h_{SN_2-AN_2}\left(\frac{1}{2};1\right) \\ = \frac{4}{5} \end{cases} \qquad (5.20)$$

From (5.19) and (5.20), we observe that the function $h_{SN_2-AN_2}$ (x) is increasing in x ∈ (0, ½) and decreasing in x ∈ (½ , 1), hence CONCAVE in x ∈ (0, 1), w > 0.

Utilizing (4.8) for M_{SN_2} (A || B; W) and M_{AN_2} (A || B; W) together with (5.20), we conclude that the inequality (5.2) is satisfied.

Now combining prepositions 3 and 4, we get the required inequality

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$$M_{SA}(\mu_A,\mu_B;W)\le\frac{4}{5}M_{SN_2}(\mu_A,\mu_B;W)$$
$$\le 4M_{AN_2}(\mu_A,\mu_B;W)$$

Preposition 5 : The following holds good :

Lower Bound for M_{SN_3}

$$M_{SA}(\mu_A(x),\mu_B(x);W)\le\frac{3}{4}M_{SN_3}(\mu_A(x),\mu_B(x);W) \quad (5.21)$$

or

$$M_{SA}(A\|B;W)\le\frac{3}{4}M_{SN_3}(A\|B;W)$$

Proof : We define

$$h_{SA-SN_3}(x)=\frac{f''_{SA}(x)}{f''_{SN_3}(x)},\qquad \forall x\in(0,1)$$

$$h_{SA-SN_3}(x)=\frac{P}{P+\frac{1}{48}\left(\frac{x}{2}\right)^{-\frac{3}{2}}+\frac{1}{48}\left(\frac{1-x}{2}\right)^{-\frac{3}{2}}} \quad (5.22)$$

Differentiating (5.22) w.r.t. x, we get

$$h'_{SA-SN_3}(x)=\frac{DP'-P\left[P'-\frac{3}{192}\left(\frac{x}{2}\right)^{-\frac{5}{2}}+\frac{3}{192}\left(\frac{1-x}{2}\right)^{-\frac{5}{2}}\right]}{D^2} \quad (5.23)$$

From (5.23), we observe that

$$h'_{SA-SN_3}(x)\begin{cases}\ge 0, when\ x\le\frac{1}{2}\\ \le 0, when\ x\ge\frac{1}{2}\\ M=\sup_{x\in(0,1)}h_{SA-SN_3}(x)\\ =h_{SA-SN_3}\left(\frac{1}{2}\right)\\ =\frac{3}{4}\end{cases} \quad (5.24)$$

From (5.23) and (5.24), we observe that the function $h_{SA-SN_3}(x)$ is increasing in x ∈ (0, ½) and decreasing in x ∈ (½ , 1), hence CONCAVE in x ∈ (0, 1), w > 0.

Applying (4.8) for M_{SA} (A ‖ B; W) and M_{SN_3} (A ‖ B; W) together with (5.24), we get the required inequality (5.21).

Preposition 6 : The following inequality holds good : Upper Bound M_{SN_3}

$$\frac{3}{4}M_{SN_3}(\mu_A(x),\mu_B(x);W)\le\frac{2}{3}M_{SN_1}(\mu_A(x),\mu_B(x);W) \quad (5.25)$$

or $$M_{SN_3}(A\|B;W)\le\frac{8}{9}M_{SN_1}(A\|B;W)$$

Proof : We define

$$h_{SN_3-SN_1}(x)=\frac{f''_{SN_3}(x)}{f''_{SN_1}(x)},\qquad \forall x\in(0,1), w>0$$

$$h_{SN_3-SN_1}(x)=\frac{P+\frac{1}{48}\left(\frac{x}{2}\right)^{-\frac{3}{2}}+\frac{1}{48}\left(\frac{1-x}{2}\right)^{-\frac{3}{2}}}{P+\frac{1}{4}(2x)^{-\frac{3}{2}}+\frac{1}{4}(2(1-x))^{-\frac{3}{2}}} \quad (5.26)$$

Differentiating (5.26) w.r.t. x, we get

$$h'_{SN_3-SN_1}(x)=\frac{D\left\{P'-\frac{3}{192}\left(\frac{x}{2}\right)^{-\frac{5}{2}}+\frac{3}{192}\left(\frac{1-x}{2}\right)^{-\frac{5}{2}}\right\}}{D^2}$$

$$-\frac{N\left\{P'-\frac{3}{4}(2x)^{-\frac{5}{2}}+\frac{3}{4}(2(1-x))^{-\frac{5}{2}}\right\}}{D^2} \quad (5.27)$$

We conclude from (5.27)

$$h'_{SN_3-SN_1}(x)\begin{cases}\ge 0, when\ x\in\left(0,\frac{1}{2}\right)\\ \le 0, when\ x\in\left(\frac{1}{2},1\right)\\ M=\sup_{x\in(0,1)}h_{SN_3-SN_1}(x)\\ =h_{SN_3-SN_1}\left(\frac{1}{2}\right)\\ =\frac{3}{4}\end{cases} \quad (5.28)$$

From (5.28), we observe that the function $h_{SN_2-SN_1}(x)$ is increasing in x ∈ (0, ½) and

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decreasing in x $\in$ (½ , 1), hence CONCAVE in $x \in (0, 1)$, $w > 0$.

Applying (4.8) for $M_{SN_3}(A \parallel B; W)$ and M_{SN_1} (A ∥ B;W) together with (5.28), we get the required inequality (5.25).

Now combining prepositions (5) and (6), we get the required inequality

$$M_{SA}(\mu_A, \mu_B; W) \le \frac{3}{4} M_{SN_3}(\mu_A, \mu_B; W) \le \frac{2}{3} M_{SN_1}(\mu_A, \mu_B; W)$$

REFERENCE [1] D. Bhandari and N. R. Pal (1992) : "Some New Information Measures for Fuzzy Sets", Information Sciences, Vol. 67, pp. 209-228. | [2] Bhatia , P.K. and Singh, S. (2013) : "A New Measure of Fuzzy Directed Divergence and its Application in Image Segmentation", I.I. Intelligent Systems and Applications, 04, pp 81-89. | [3] G. J. Klir and T. A. Folgev (1995) : "Fuzzy Sets, Uncertainty and Information", Printice Hall of India, Pvt. Ltd., New Delhi. | [4] G. Jia and J.D. Cao (2003), "A New Upper Bound of the Logarithmic Mean" J. Ineq. Pure & Appl. Math. 4(4), Article 80: http://jipam.vu.edu.au/v 4 n 4/088-03.pdf. | [5] H.N. Shi, J. Zhang and Da-Moa Li (2010) : "Schur-Geometric convexity for difference of means, Applied Mathematics E-notes, 10(2010), 275-284. | [6] H. N. Shi and Zia Zhang, (2010) : "Schur-Geometric concavity for difference of means available on line – http:/rgmia.org/paper/s/v 12 n u /huannan2.pdf.2010. | [7] I., Csiszar, (1967 a) : "Information Type Measures of Difference of Probability Distribution and Indirect Observation", Studia Math, Hungrica, 2, pp. 299-318. | [8] I. J. Taneja (2005) : "Refinement of Inequalities Among Means", ar X iv : Math GM 10505192 V 2, 12 June 2005. | [9] J. Ch. Kuang, (1993), "Applied Inequalities", Hunan Education Press, 2nd Ed. 1993 (Chinees). | [10] Javier E. Contreras – Reyes and Reinaldo B. Arellano-Valle (2012) : " Kullback-Leibler Divergence Measure for Multivariate Skew-Normal Distributions", Entropy, 14, 1606-1626 : doi: 10.3390/& 14091606. | [11] Kapur, J. N. (1997) : "Measures of Fuzzy Information" M.S.T.S, New Delhi. | [12] Kapur and Sharma (1998): "M-Entropy measures And Their Application", Ph.D. Thesis, pp. 46-54. | [13] Kullback-Leibler (1951) : "On Information and Sufficiency", An Math. Stat.22, 79-86,. | [14] Ladislav Mate Jicka (2013) : "Bounds for the convex combination of the first Seiffert and Logarithmic Means in terms of Generalized Heronian Mean" Published in British Journal of Mathematics & Computer Science, 3(3) : 265-274, (2013). | [15] Liu H. Meng X-J(2011) : "The Optimal Convex Combination Bounds or Seiffert's Mean" J. Inequalities, , Art.ID 686834, 9 pp. | [16] Nikhil R. Pal and James C. Bezdek (1994) : Measuring Fuzzy Uncertainty" IEEE Transactions on Fuzzy Systems, Vol. 2, No. 2. May (1994), pp. 107-118. | [17] Parkash, O. (2000) : "On Fuzzy Symmetric Divergence" AFSS, pp. 904-908. | [18] Priti, Sharma and Singh (2011) : "On concavity of Difference-Divergence measures of Fuzzy Information I, II" I. Indian Soc. Stat. Opers. Res. Vol. XXXII No. 1-4 December, 2011, pp. 17-32, 33-48. | [19] Priti Gupta, Santhosh Kumari and RP Singh (2012) : "Inequalities Among Weighted Fuzzy Mean Difference-Measures and Their Concavity", Aryabhatta Journal of Mathematics and Informatics, Vol. 4 No. 2, July-Dec. (2012) pp. 217-34. | [20] Singh, R. P. and V. P. Tomar (2008): "On Fuzzy Divergence measures and their inequalities", Published in the Proceedings of 10th national Conference of ISITA, pp. 41-43, (2008). | [21] Zh. G. Xiao and Zh.-H. Zhang (2003), "The Inequalities G ⭘ L ⭘ I ⭘ A in n variables", I. Ineq. Pure and Appl. Math., 4(2); Article 39, http://jipamovu.edu.au/v 4 n 2/ 110-02.pfd . | [22] Zhi-Hua Zhang and Yu-Dong Wu, (2004): "The new bounds of the Logarithmic Mean, RGMIA Research report collection, http://rgmia.vu.edu.av. |