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Some Fixed Point Theorems in Menger Space Using Weak Contractive Mappings

KEYWORDS: Menger space, Weak contraction, fixed point

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ABSTRACT *The aim of the present paper is to establish fixed point theorems for self mappings under weak contractions in Menger space.*

1. INTRODUCTION

The study of fixed points of mappings in a Menger space satisfying certain contractive conditions has been at the center of vigorous research activity. The concept of Menger space was introduced by K. Menger in 1942. Sehal and Bharucha Reid first introduced the contraction mapping principle in probabilistic metric space. Hadzic and Pap has given a comprehensive survey of this line research.

An altering distance function is a control function which alters the distance between two points in a metric space. This concept was introduced by Khan, Swaleha and Sessa [7]. Recently altering distance functions have been extended in the context of Menger space by Choudhury and Das [1]. This idea of control function in Menger space has opened the possibility of proving new probabilistic fixed point results.

Recently, Rhoades [10] proved interesting fixed point theorems for ψ - weak contraction in complete metric space. The significance of this kind of contraction can also be derived from the fact that they are strictly relative to famous Banach's fixed point theorems and to some other significant results. Also, motivated by the results of Rhoades and on the lines of Khan *et. al.* employing the idea of altering distances, Vetro *et.al.* [13] extended the notion of (ϕ,ψ)- weak contraction to fuzzy metric space and proved common fixed point theorem in fuzzy metric space.

The purpose of this paper is to extend the weak ϕ- contraction and (ϕ,ψ)- contraction in Menger space and to obtain fixed point theorems for self mappings satisfying these weak contractive conditions.

2. PRELIMINARY NOTES

Here we recall the definitions, examples and results which will be used in the following section.

Definition 2.1. A mapping $F : R \to R^+$ is called a distribution if it is non-decreasing left continuous with $\inf \{F(t) : t \in R\} = 0$ and $\sup\{F(t) : t \in R\} = 1$:

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, t \leq 0 \\ 1, t \leq 0 \end{cases}$$

Definition 2.2. A probabilistic metric space (PM-space) is an ordered pair (X, F) where X is an arbitrary set of elements and $F : X \times X \to L$ (written as (p, q) $\mapsto F_{p,q}$), is subjected to the conditions

i) $F_{p,q}(x) = 1$ for all $x > 0$, if and only if $p = q$;
ii) $F_{p,q}(0) = 0$;
iii) $F_{p,q} = F_{q,p}$
iv) If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$ then $F_{p,r}(x + y) = 1$.

Definition 2.3. A mapping $t: [0, 1] \times [0, 1] \to [0,1]$ is called a t- norm if

i) $t(a, 1) = a$, $t(0, 0) = 0$;
ii) $t(a, b) = t(b, a)$;
iii) $t(c, d) \leq t(a, b)$ for $c \leq a$, $d \leq b$;
iv) $t(t(a, b), c) = t(a, t(b, c))$.

Definition 2.4. A Menger space is a triplet (X, F, t) where (X, F) is a PM-space and t is a t-norm such that for all p, q, r ∈ X and for all x, y ≥ 0

$F_{p,r}(x + y) \geq t(F_{p,q}(x), F_{q,r}(y))$

Example 2.5. If (X,d) is a metric space then the metric d induces a mapping $X \times X \to L$, defined by $F_{p,q}(x) = H(x- d(p, q))$, p, q ∈ X and x ∈ R. Further, if the t- norm $t : [0, 1] \times [0, 1] \to [0, 1]$ is defined by $t(a, b) = \min \{a, b\}$, then (X, F, t) is a Menger space. The space (X, F, t) so obtained is called the induced Menger space.

Example 2.6. Let (X, d) be a metric space. We define the t-norm $*$ by $a * b = ab$ for all a,b ∈ [0, 1]. We define the distribution function F as

$$F_{x,y}(t)=\begin{cases} \frac{t}{t+d(x,y)} & if\ t > 0 \\ 0 & if\ t \leq 0 \end{cases} \text{ for all } x, y \in X.$$

Then (X, F, $*$) is a Menger space.

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Definition 2.7. Let (X, F, $*$) be a Menger space. Then

i) A sequence $\{x_n\}$ in X is said to be a G-Cauchy sequence if and only if $\lim_{n\to+\infty} F_{x_{n+p},x_n}(t) = 1$ for any p > 0 and t > 0.

ii) The Menger space (X, F, $*$) is called G - complete if every G – Cauchy sequence in X is convergent.

For convenience, we denote by Ƒ the class of functions ϕ : [0, ∞) → [0, ∞) non- decreasing and continuous satisfying ϕ (t) > 0 for t ∈ (0, ∞) and ϕ (0) = 0.

3. MAIN RESULTS

Definition 3.1. Let (X, F, $*$) be a Menger space with $F_{x,y}(t) > 0$ for all t > 0 and x, y ∈ X. An operator f : Y → Y is called a weak ϕ -contraction if $\left(\frac{1}{F_{fx,fy}(t)}-1\right) \le \left(\frac{1}{F_{x,y}(t)}-1\right) - \phi\left(\frac{1}{F_{x,y}(t)}-1\right)$ where $\phi \in$ Ƒ.

Theorem 3.2. Let (X, F, $*$) be a G-complete Menger space with $F_{x,y}(t) > 0$ for all t > 0 and x, y ∈ X. Let f : X → X be a weak ϕ contraction. Then f has a unique fixed point.

Proof:

Construct a sequence $\{x_n\}$ in X such that $x_{n+1} = f(x_n)$ for all n ∈ N.
Assume $x_{n+1} \neq x_n$ for all n ∈ N. (Otherwise f has a fixed point)
Using the contractive condition,

$$\left(\frac{1}{F_{x_n,x_{n+1}}(t)} - 1\right) = \left(\frac{1}{F_{fx_{n-1},fx_n}(t)} - 1\right)$$

$$\le \left(\frac{1}{F_{x_{n-1},x_n}(t)} - 1\right) - \phi\left(\frac{1}{F_{x_{n-1},x_n}(t)} - 1\right) \quad (1)$$

$$< \left(\frac{1}{F_{x_{n-1},x_n}(t)} - 1\right)$$

This implies that $F_{x_n,x_{n+1}}(t) > F_{x_{n-1},x_n}(t)$ for all $n \ge 1$ and so $\{F_{x_{n-1},x_n}(t)\}$ is a non-decreasing sequence of positive real numbers in (0, 1).

If possible let S (t) = $\lim_{n\to+\infty} F_{x_{n-1},x_n}(t) < 1$ for some t > 0.
Letting n → ∞ in (1) we obtain,

$$\frac{1}{S(t)} - 1 \le \left(\frac{1}{S(t)} - 1\right) - \phi\left(\frac{1}{S(t)} - 1\right)$$

which is a contradiction.
Therefore S(t) = 1 for all t > 0.
That is, $F_{x_{n-1},x_n}(t) \to 1$ as n → ∞.
Now for each positive integer p, we have

$$F_{x_n,x_{n+p}}(t) \le F_{x_n,x_{n+1}}(t/p) * F_{x_{n+1},x_{n+2}}(t/p)$$

$$* \cdots * F_{x_{n+p-1},x_{n+p}}(t/p)$$

It follows that

$$\lim_{n\to+\infty} F_{x_n,x_{n+p}}(t) \ge 1 * 1 * \cdots * 1 = 1$$

This implies that $\{x_n\}_{n\ge0}$ is a G- Cauchy sequence in X.
Now, the G- completeness of X gives an element x ∈ X such that $x_n \to x$.
Using the contractive condition,

$$\left(\frac{1}{F_{x_{n+1},fx}(t)} - 1\right) = \left(\frac{1}{F_{fx_n,fx}(t)} - 1\right)$$

$$\le \left(\frac{1}{F_{x_n,x}(t)} - 1\right) - \phi\left(\frac{1}{F_{x_n,x}(t)} - 1\right)$$

Letting k → ∞, the above inequality yields $F_{x,fx}(t) = 1$ and hence $fx = x$.
Next we establish the uniqueness of fixed point. Let x and y be two fixed points of f.
Then

$$\left(\frac{1}{F_{x,y}(t)} - 1\right) = \left(\frac{1}{F_{fx,fy}(t)} - 1\right)$$

$$\le \left(\frac{1}{F_{x,y}(t)} - 1\right) - \phi\left(\frac{1}{F_{x,y}(t)} - 1\right)$$

$$< \left(\frac{1}{F_{x,y}(t)} - 1\right)$$

The above contradiction implies that the fixed point of f is unique.

Definition 3.3. Let (X, F, $*$) be a Menger space with $F_{x,y}(t) >$ 0 for all t > 0 and x, y ∈ X. An operator f : Y → Y is called a weak (ϕ,ψ) -contraction if $\phi\left(\frac{1}{F_{fx,fy}(t)}-1\right) \le \phi\left(\frac{1}{F_{x,y}(t)}-1\right) - \psi\left(\frac{1}{F_{x,y}(t)}-1\right)$
where $\phi \in$ Ƒ.

Theorem 3.4. Let (X, F, $*$) be a G-complete Menger space with $F_{x,y}(t) >$ 0 for all t > 0 and x, y ∈ X. Let f : X → X be a weak (ϕ,ψ) contraction. Then f has a unique fixed point.

Proof:

Construct a sequence $\{x_n\}$ in X such that $x_{n+1} = f(x_n)$ for all n ∈ N.

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Assume $x_{n+1} \neq x_n$ for all n ∈ N. (Otherwise f has a fixed point)
Using the contractive condition,

$$\phi\left(\frac{1}{F_{x_n,x_{n+1}}(t)}-1\right)=\phi\left(\frac{1}{F_{fx_{n-1},fx_n}(t)}-1\right)$$
$$\leq \phi\left(\frac{1}{F_{x_{n-1},x_n}(t)}-1\right)-\psi\left(\frac{1}{F_{x_{n-1},x_n}(t)}-1\right) \quad (2)$$
$$<\phi\left(\frac{1}{F_{x_{n-1},x_n}(t)}-1\right)$$

Since ϕ is non decreasing, we obtain that $F_{x_n, x_{n+1}}(t) > F_{x_{n-1}, x_n}(t)$ for all $n \geq 1$ and so $\{F_{x_{n-1}, x_n}(t)\}$ is a non-decreasing sequence of positive real numbers in (0, 1).

If possible let $S(t) = \lim_{n\to+\infty} F_{x_{n-1}, x_n}(t) < 1$ for some t > 0.
Letting n → ∞ in (2) we obtain,

$$\phi\left(\frac{1}{S(t)}-1\right) \leq \phi\left(\frac{1}{S(t)}-1\right)-\psi\left(\frac{1}{S(t)}-1\right)$$
$$<\phi\left(\frac{1}{S(t)}-1\right)$$

which is a contradiction.
Therefore S(t) = 1 for all t > 0.
That is, $F_{x_{n-1}, x_n}(t) \to 1$ as n → ∞.
Now for each positive integer p, we have

$$F_{x_n, x_{n+p}}(t) \leq F_{x_n, x_{n+1}}(t/p) * F_{x_{n+1}, x_{n+2}}(t/p) * \cdots * F_{x_{n+p-1}, x_{n+p}}(t/p)$$

It follows that

$$\lim_{n\to+\infty} F_{x_n, x_{n+p}}(t) \geq 1 * 1 * \cdots * 1 = 1$$

This implies that $\{x_n\}_{n\geq 0}$ is a G- Cauchy sequence in X.
Now, the G- completeness of X gives an element x ∈ X such that $x_n \to x$.
Using the contractive condition,

$$\phi\left(\frac{1}{F_{x_{n+1},fx}(t)}-1\right)=\phi\left(\frac{1}{F_{fx_n,fx}(t)}-1\right)$$
$$\leq \phi\left(\frac{1}{F_{x_n,x}(t)}-1\right)-\psi\left(\frac{1}{F_{x_n,x}(t)}-1\right)$$

Letting k → ∞, the above inequality yields $F_{x,fx}(t) = 1$ and hence $fx = x$.
Next we establish the uniqueness of fixed point. Let x and y be two fixed points of f.
Then

$$\phi\left(\frac{1}{F_{x,y}(t)}-1\right)=\phi\left(\frac{1}{F_{fx,fy}(t)}-1\right)$$
$$\leq \phi\left(\frac{1}{F_{x,y}(t)}-1\right)-\psi\left(\frac{1}{F_{x,y}(t)}-1\right)$$
$$<\phi\left(\frac{1}{F_{x,y}(t)}-1\right)$$

Since ϕ is non-decreasing the above contradiction imply that the fixed point of f is unique.

If in the above Theorem we take ϕ as the identity mapping on [0, ∞) which we denote by $Id_{[0,\infty)}$, we obtain the following Corollary which is nothing but Theorem 3.2

Corollary 3.5. Let (X, F, ∗) be a G-complete Menger space with $F_{x,y}(t) > 0$ for all t > 0 and x, y ∈ X. Let f : X → X be a weak ($Id_{[0,\infty)}$, ψ) - contraction, ψ ∈ Ƒ. Then f has a unique fixed point.

Now we obtain the common fixed point theorem using weak contraction on two mappings f; g with the additional condition that the pair (f; g) is weakly compatible.

Definition 3.6. Let (X, F, ∗) be a Menger space with $F_{x,y}(t) > 0$ for all t > 0 and x, y ∈ X. Two operators f, g : X → X are called a weak (ϕ,ψ) contraction if

$$\phi\left(\frac{1}{F_{fx,fy}(t)}-1\right)\leq \phi\left(\frac{1}{F_{gx,gy}(t)}-1\right)-\psi\left(\frac{1}{F_{gx,gy}(t)}-1\right)$$

for any x ∈ A_i, y ∈ A_{i+1}; i = 1,2,…m, where $A_{m+1} = A_1$ and $\phi,\psi \in$ Ƒ.

Theorem 3.7. Let (X, F, ∗) be a G-complete Menger space with $F_{x,y}(t) > 0$ for all t > 0 and x, y ∈ X. Let f, g : X → X be two weak (ϕ,ψ) contraction such that g is onto. If the pair (f, g) is weakly compatible then f and g have a common fixed point. Furthermore, if g is one to one then f and g have a unique common fixed point in X.

Proof: Let x_1 be an arbitrary point in X. Since g is onto, for this point x_1, there exists x_2 in X such that $fx_1 = gx_2$.
Inductively, one can construct a sequence $\{x_n\}$ such that $fx_n = g\,x_{n+1}$ for n = 1, 2, … .
Next we prove $\{gx_n\}$ is a G-Cauchy sequence.
If there exists n_0 ∈ N such that $gx_{n_0+1} = gx_{n_0}$ then $x_{n_0} = fx_{n_0}$, that is, x_{n_0} appears as a common fixed point of f and g.
Suppose that $gx_{n+1} \neq gx_n$ for all n.

Since f and g are weak (ϕ,ψ) contraction, we have

$$\phi\left(\frac{1}{F_{gx_{n+1},gx_n}(t)}-1\right)=\phi\left(\frac{1}{F_{fx_n,fx_{n-1}}(t)}-1\right)$$

$$\leq\phi\left(\frac{1}{F_{gx_n,gx_{n-1}}(t)}-1\right)-\psi\left(\frac{1}{F_{gx_n,gx_{n-1}}(t)}-1\right) \quad (3)$$

$$<\phi\left(\frac{1}{F_{gx_n,gx_{n-1}}(t)}-1\right)$$

Since ϕ is non decreasing, we get $F_{gx_n,gx_{n+1}}(t) > F_{gx_{n-1},gx_n}(t)$ for all n for all n.
Hence the sequence $\{F_{gx_n,gx_{n-1}}(t)\}$ is a non-decreasing sequence of positive real numbers in (0, 1].
Let S (t) = $\lim_{n\to+\infty} F_{gx_n,gx_{n-1}}(t)$.
Next we show that S(t) = 1 for all t > 0.
If not there exists some t > 0 such that S(t) < 1.
Then, on making n → ∞ in (3) we obtain

$$\phi\left(\frac{1}{S(t)}-1\right)\leq\phi\left(\frac{1}{S(t)}-1\right)-\psi\left(\frac{1}{S(t)}-1\right)$$

which is a contradiction.
Therefore S(t) = 1 for all t > 0.
That is, $F_{gx_n,gx_{n-1}}(t)\to 1$ as n → ∞.
So that for each positive integer p, we have

$$F_{gx_n,gx_{n+p}}(t)\leq F_{gx_n,gx_{n+1}}(t/p)*F_{gx_{n+1},gx_{n+2}}(t/p)$$

$$*\cdots*F_{gx_{n+p-1},gx_{n+p}}(t/p)$$

It follows that

$$\lim_{n\to+\infty}F_{gx_n,gx_{n+p}}(t)\geq 1*1*\cdots*1=1$$

So that $\{gx_n\}$ is a G-Cauchy sequence in X. Since X is complete, $\{gx_n\}$ converges to x for some $x \in$ X. Since g is onto for this x there exists $z \in$ X such that

$$\lim_{n\to\infty} gx_n = x = gz$$

Using the contractive condition, we obtain

$$\phi\left(\frac{1}{F_{gx_{n+1},gx_n}(t)}-1\right)=\phi\left(\frac{1}{F_{fx_n,fz}(t)}-1\right)$$

$$\leq\phi\left(\frac{1}{F_{gx_n,gz}(t)}-1\right)-\psi\left(\frac{1}{F_{gx_n,gz}(t)}-1\right)$$

Letting n → ∞ in the above inequality, we obtain $F_{gz,fz}(t)$ = 1 and hence *fz = gz = x.*
Since f and *g* are weakly compatible and *fz = gz, ffz = fgz =gfz = ggz*. That is *fx = gx.*
Using the contractive condition we obtain,

$$\phi\left(\frac{1}{F_{fz,ffz}(t)}-1\right)\leq\phi\left(\frac{1}{F_{gz,gfz}(t)}-1\right)-\psi\left(\frac{1}{F_{gz,gfz}(t)}-1\right)$$

$$\leq\phi\left(\frac{1}{F_{gz,gfz}(t)}-1\right)$$

$$=\phi\left(\frac{1}{F_{fz,ffz}(t)}-1\right)$$

From the last inequality we have, $\psi\left(\frac{1}{F_{gz,gfz}(t)}-1\right)=0$

Since $\psi \in$ F, $F_{gz,gfz}(t)$ = 1. This implies gz = gfz and consequently x = fz = gz = gfz = gx = fx.
That is, fx = gx = x. Thus x is a common fixed point of f and g.
To prove the uniqueness of fixed point, let us assume $y, z \in \bigcap_{i=1}^{m} A_i$ be two distinct common fixed points of f and g.

$$\phi\left(\frac{1}{F_{y,z}(t)}-1\right)=\phi\left(\frac{1}{F_{fy,fz}(t)}-1\right)$$

$$\leq\phi\left(\frac{1}{F_{gy,gz}(t)}-1\right)-\psi\left(\frac{1}{F_{gy,gz}(t)}-1\right)$$

$$<\phi\left(\frac{1}{F_{gy,gz}(t)}-1\right) \quad \text{Since g is one to one}$$

$$=\phi\left(\frac{1}{F_{y,z}(t)}-1\right) \quad \text{, which is a contradiction}$$

Therefore y = z.
Hence the common fixed point is unique.

Corollary 3.8. Let (X, F, ∗) be a G-complete Menger space with $F_{x,y}(t) > 0$ for all t > 0 and x, y ∈ X. Let f : X → X be a weak (ϕ,ψ) contraction. Then f has a unique fixed point.

The proof follows by taking g as the identity function in the above theorem.

Corollary 3.9. Let (X, F, ∗) be a G-complete Menger space with $F_{x,y}(t) > 0$ forall t > 0 and x, y ∈ X. Let f : X → X be a weak ψ contraction. Then f has a unique fixed point.

Proof can be easily obtained by setting ϕ as the identity function in the above corollary.

The following example illustrates Theorem 3.2

Example 3.10. Let X = [0, 1] and the distribution function F be defined as

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$$F_{x,y}(t) = \begin{cases} \frac{t}{t+d(x,y)} & if\ t > 0 \\ 0 & if\ t = 0 \end{cases}$$

For all x, y $\in$ X. Define the t-norm $*$ as a$*$b = ab. Then (X, F, $*$) is a complete Menger space.
Define f : X → X by fx = $x/2$. Also define $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = t/2$.

Then f satisfies the condition of Theorem 3.2 and f has a unique fixed point 0 $\in$ X.

The following example illustrates Theorem 3:4

Example 3.11. Let X = [0, 1] and the distribution function F be defined as

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+d(x,y)} & if\ t > 0 \\ 0 & if\ t = 0 \end{cases}$$

For all x, y $\in$ X. Define the t-norm $*$ as a$*$b = ab. Then (X, F, $*$) is a complete Menger space.
Define f : X → X by fx = $x/2$. Also define $\phi, \psi : [0,\infty) \to [0,\infty)$ by $\phi(t) = t/2$ and $\psi(t) = t/3$

Then the condition of Theorem 3.4 is satisfied by f and f has a unique fixed point 0 $\in$ X.

The next example illustrates Theorem 3.7.

Example 3.12. Let X = [0, 1] and the distribution function F be defined as

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+d(x,y)} & if\ t > 0 \\ 0 & if\ t = 0 \end{cases}$$

For all x, y $\in$ X. Define the t-norm $*$ as a$*$b = ab. Then (X, F, $*$) is a complete Menger space.
Define f; g : X → X such that f x = $x^2/12$ and g x = $x/3$. Suppose that $\phi, \psi : [0,\infty) \to [0,\infty)$ by $\phi(t) = t/2$ and $\psi(t) = t/4$

Then all the conditions of Theorem 3.7 are satisfied and 0 is the unique common fixed point of f and g.

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