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# Banach Contraction principle in Generalized Q-fuzzy metric space

**KEYWORDS** Q-fuzzy metric space, Banach contraction principle, Convegence, Cauchy sequences.

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**ABSTRACT** *An extended version of Banach Contraction principle in the framework of Q-fuzzy metric space is obtained in this paper.*

**Introduction**

G.Sun et.al [2] introduced the notion called Q-fuzzy metric space which is a generalization of fuzzy metric space. Properties of Q-fuzzy metric space and fixed point theorems in Q-fuzzy metric space can be seen in [1]. The results in [1] modified the results due to Sushil Sharma[5] and Rhoades[4]. In [3] common coupled fixed point theorems for self maps in symmetric Q-fuzzy metric space is obtained.

**Preliminaries**

G.sun et.al [2] defined the Q-fuzzy metric space as follows:

**Definition 2.1 :** A 3-tuple $(X,Q,*)$ is called a Q-fuzzy metric space if X is an arbitrary set, $*$ is a continuous t norm and Q is a fuzzy st on $X^3\times(0,\infty)$ satisfying the following conditions, for each $x,y,z,a\in X$ and $t,s>0$.

- $Q(x,x,y,t)>0$ and $Q(x,x,y,t)\geq Q(x, y,z,t)$ for all $x,y,z\in X$ with $z\neq y$.
- $Q(x, y,z,t)=Q(P\{x,y,z\},t)$ (Symmetry) where P is a permutation function.
- $Q(x, y,z,t)=1$ if and only if $x=y=z$.
- $Q(x, y,z,t+s)\geq Q(x,a,a,t)*Q(a,y,z,s)$.
- $Q(x, y,z,.) : (0,\infty) \to[0,1]$ is continuous.

**Definition 2.2 :** A Q-fuzzy metric space $(X,Q,*)$ is said to be symmetric if $Q(x,y, y, t) = Q(x, x, y, t)$ for all $x, y \in X$.

**Definition 2.3 :** A sequence $(x_n)$ in X converges to x if and only if $Q(x_n, x_n ,x,t) \to 1$ as $n \to 1$, for each $t > 0$.

**Definition 2.4 :** A sequence $(x_n)$ in X is called a Cauchy sequence if for each $0 < \varepsilon< 1$ and $t > 0$, there exists an $n_0 \in N$ such that $Q(x_m, x_n, x_n, t) > 1 -\varepsilon$ for each $m, n \geq n_0$.

The following two results were obtained by G. Sun et.al.[2].

**Lemma 2.1 :** Let $(X,Q,*)$ be a Q-fuzzy metric space, then $Q(x,y,z, t)$ is non-decreasing with respect to t for all $x,y,z \in X$.

**Lemma 2.2 :** Let $(X,Q,*)$ be a Q-fuzzy metric space. Then if

a) there exists a $k \in (0, 1)$ such that :
$Q(y_{n+2},y_{n+1}, y_{n+1}, kt) \geq Q(y_{n+1},y_n,y_n, t)$
for each $t > 0$ and $n \in N$. Then $\{y_n\}$ is a Cauchy sequence in X.

b) there exists a $k \in (0, 1)$ such that :
$Q(x,y,z,kt) \geq Q(x,y,z,t)$ for each $t > 0$ and $n \in N$.

Then $x = y = z$.

**Definition 2.5 :** Let $(X,Q,*)$ be a Q-fuzzy metric space. Then Q is a continuous function on $X^3\times(0,\infty)$.

**Main Results**

Followed by the classical Banach contraction principle, fuzzy version of Banach contraction theorem was studied by Grabiec[1] in 1988. Motivated by Grabiec's result, the Q-fuzzy version of Banach's fixed point theorem is as follows:

**Theorem 3.1:** Let $(X,Q,*)$ be a complete Q-fuzzy metric space such that for all $x, y \in X$. Let $T : X \to X$ be a mapping satisfying $Q(Tx, Ty, Ty, kt) \geq Q(x,y, y, t)$ for all $x, y \in X$, $0 < k < 1$. Then T has a unique fixed point.

**Proof :**

Let $x \in X$ and $x_n=T^nx$, $n \in N$.

$$\begin{aligned} Q(x_n,x_{n+1}, x_{n+1}, kt) &= Q(T^nx, T^{n+1}x, T^{n+1}x,kt) \\ &\geq Q(T^{n-1}x, T^nx, T^nx,t) \\ &= Q(x_{n-1},x_n, x_n, t) \end{aligned}$$

Also,

$$\begin{aligned} Q(x_n,x_{n+1}, x_{n+1}, t) &\geq Q(x_{n-1},x_n, x_n, ) \\ &\geq Q(x_{n-2},x_{n-1}, x_{n-1},) \\ &\geq Q(x,x_1, x_1, ) \end{aligned}$$

$$Q(x_n,x_{n+p}, x_{n+p}, t) \geq Q(x_n,x_{n+1}, x_{n+1}, )$$

$$\begin{aligned} &* Q(x_{n+1},x_{n+p}, x_{n+p}, ) \\ &\geq Q(x_n,x_{n+1}, x_{n+1}, ) \\ &* Q(x_{n+1},x_{n+2}, x_{n+2}, ) \\ &* Q(x_{n+2},x_{n+p}, x_{n+p}, ) \\ &\geq Q(x_n,x_{n+1}, x_{n+1}, ) \\ &* Q(x_{n+1},x_{n+2}, x_{n+2}, ) \\ &* Q(x_{n+p-1},x_{n+p}, x_{n+p}, ) \end{aligned}$$

Taking limit, we get $\{x_n\}$ as Cauchy. Since X is complete $x_n\to y$ in X.

$$\begin{aligned} Q(Ty, y, y, t) &\geq Q(Ty,Tx_n,Tx_n,)* Q(Tx_n,y,y, ) \\ &\geq Q(y,x_n,x_n,)* Q(x_{n+1},y,y, ) \end{aligned}$$

This converges to 1 as $n\to\infty$. This implies $Ty=y$.
Next to show the uniqueness.
$1\geq Q(z,y,y,t) = Q(Tz,Ty,Ty,t)$
$\geq Q(z,y,y,) = Q(Tz,Ty,Ty,)$
$\geq Q(z,y,y,)\geq\geq Q(z,y,y,)\to 1$

Thus $Q(z,y,y,t) =1$. Which implies $z=y$.

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