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Simplicity of Group Rings and Ideals of RG

KEYWORDS

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ABSTRACT *The group ring RG of a group G over a commutative unital ring R is an interesting object of study for both group theory and ring theory [4]. Tahara [3] defines on integral group ring ZG to be simple if and only if I is an ideal of ZG, then G (1+I)=1 or G. We propose to call such a group ring Tahara-simple. We generalize this definition.*

Definition 1: Let R be a commutative ring with unity. Then the group ring RG is Tahara – simple iff the following condition holds :

If I is an ideal of RG then $G\cap(1+I)=1$ or G.

We prove the followings :

Theorem 1.1: Let G be a group. Then G is simple if and only if the group ring RG is Tahara-Simple.

Proof : Let G be a simple group. Then for any ideal I of RG, $G\cap(1+I)$ is a normal subgroup of G and hence $G\cap(1+I)=1$ or G conversely assume that if I is any ideal of RG, then $G\cap(1+I)=1$ or G. let $N\neq 1$ be normal subgroup of G.

We consider a canonical ring homomorphism

$$\overline{f} : RG \rightarrow R\,(G/N)$$

Then ker $\overline{f}$ = RG (N–1) is an ideal of RG, and hence by the assumption, we have N = $G\cap$ (1+RG. (N–1) = 1 or G and therefore N=G. Thus G is simple.

Next we can determine all ideals I of RG with property

$G\cap(1+I)= G$ as follows.

Theorem 1.2 : Let R be a principal ideal domain and I be any ideal of RG then $G\cap(1+I)=G$ if and only if there exists an element $x\in R$ such that.

$I = \Delta(G)+xR$, where $\Delta(G)$ is the augmentation ideal of RG.

Proof : Assume that I is an ideal of RG with $G\cap(1+I)=G$. Let g be an element of G. Then $g\in G= G\cap(1+I)$ and hence $g-1\in I$. Thus $\Delta(G)$ is contained in I

We consider the quotient ring.

$RG/\Delta(G) \supseteq I/\Delta(G)$. Here $RG/\Delta(G)$ is isomorphic to R, and hence $I/\Delta(G)$ is identified with some ideal of R say J. Since R is a principal ideal domain,

$\therefore$ J is a principal ideal and hence $\exists$ an element $x\in R$ such that $J=\langle x\rangle$.

Since I/ Δ(G) is identified with J, $\therefore$ I is equal to Δ(G)+xR.

Conversely assume that I = Δ(G)+xR for some $x \in R$

Then $G \cap (1+I) = G \cap (1+\Delta(G)+xR)$

$\supseteq G \cap (1+\Delta(G))$

= G

and hence $G \cap (1+I) = G$

From theorems [1.1] and [1.2], In order to determine if a given group G is simple or not we have to characterize all ideal I with property $G \cap (1+I)=1$. Thus we have the following problem :

Problem : Characterization of all ideal I of RG, with property $G \cap (1+I)=1$

Next we recall the definition of a simple ring.

Definition 2 : ([1,2]) A ring R is simple if and only if for any ideal I of R, I=0 or R. That is the set of all ideals of R is equal to {0, R}.

Now we propose the following definition for the simplicity of the group ring RG.

Definition 3 : The group ring RG is simple if and only if for any ideal I of RG, $G \cap (1+I)=1$ or G, that is the set of all ideal of RG is equal to $\{\Delta(G)+xR,\ x \in R\}$. In other words, the group ring RG is simple if and only if the group G is simple.

Thus in order to classify simple group rings we need to characterize all ideals I of RG with the property $G \cap (1+I)=1$ or G.

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