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LEFT-STAR AND RIGHT-STAR PARTIAL ORDERING OF S-UNITARY MATRICES

KEYWORDS: left-star partial ordering , right-star partial ordering, star partial ordering, s-unitary matrix.

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ABSTRACT *Some results relating to left-star,right-star partial ordering of s-unitary matrices are obtained. Based on these results ,theorems relating to several characterizations of left-star partial ordering between right-star partial ordering of s-unitary matrices are derived.*

1.INTRODUCTION

The relationship between the star and minus partial ordering was settled by Baksalary [3] . Hartwig and Styan [4] developed some characterization of the star partial ordering for matrices and rank subtractivity. Jurgen Grob observed some remarks on partial ordering of hermitian matrices in [7].Liu and Yang [11] have shown some results on the partial ordering of block matrices.In [6] Jorma.K.Merikoski and Xiaogi Liu have developed star partial ordering on normal matrices. Krishnamoorthy and Govindarasu derived theorems relating to lowener partial ordering and star partial ordering [10].The concept of 'θ' partial ordering [Theta partial ordering] of s-unitary matrices was also introduced by Krishnamoorthy and Govindarasu [9].Some characterizations of the left-star,right-star and star partial ordering between matrices of the same size are obtained Hongxing Wang and Jin Xu[5].

2.PRELIMINARIES

Let C_{nxn} be the space of nxn complex matrices of order n. Let $A^T, \overline{A}, A^*, A^S, \overline{A}^S (= A^\theta)$ denote transpose, conjugate, conjugate transpose, secondary transpose, conjugate secondary transpose of a matrix A respectively. Anna Lee[1] has initiated the study of secondary symmetric matrices. Also she has shown that for a complex matrix A, the usual transpose $A^T A^T$ and secondary transpose A^S are related as $A^S = VA^TV$ $A^S = VA^TV$ where 'V'is the associated permutation matrix whose elements on the secondary diagonal are 1 and other elements are zero. .Also $\overline{A}^S$ denotes the conjugate secondary transpose of A. i.e $\overline{A}^S$ = (c_{ij}) where c_{ij} = $\overline{a_{n-j+1,\ n-i+1}}$ [2] and $\overline{A}^S = VA^*V = A^\theta$. Also 'V' satisfies the following properties. $V^T = V^\theta = \overline{V} = V^* = V$ and $V^2 = I$.

Definition 2.1

A matrix $A \in C_{nxn}$ is said to be s-unitary(Secondary unitary) if $A^\theta A = AA^\theta = I$ [8].

Definition 2.2 Star partial ordering

Let ü, ∈ ü ,

$A \underset{*}{\leq} B$ iff $A^*A = A^*B$ and $AA^* = BA^*$.

3.LEFT-STAR AND RIGHT -STAR PARTIAL ORDERING

Definition 3.1 Left-Star partial ordering

Let $A, B \in C_{nxn}$,

$A* \leq B$ iff $A^*A = A^*B$ and $R(\mathrm{A}) \subseteq \mathrm{R(B)}$.

Definition 3.2 Right -Star partial ordering

Let $A, B \in C_{nxn}$, $A \leq *B$ iff $AA^* = BA^*$ and $R(\mathrm{A}^*) \subseteq \mathrm{R}(\mathrm{B}^*)$

Theorem 3.3 Let $AV* \leq VA$ **.If** A **is unitary then** A **is s-unitary.**

Proof : Let $AV* \leq VA$.

$AV* \leq VA \Rightarrow (AV)^* AV = (AV)^* VA$ and

$R(AV) \subseteq \mathrm{R}(VA)$

$\Rightarrow V^*A^*AV = V^*A^*VA$

$\Rightarrow VIV = VA^*VA$ [since A is unitary.]

$\Rightarrow I = VA^*VA$

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$\Rightarrow A^{-1} = VA^*V$
$\Rightarrow A^{-1} = A^\theta$

Therefore A is s-unitary.

Theorem 3.4 Let $VA* \leq AV$ .If A is unitary then A is s-unitary.

Proof: Let $VA* \leq AV$

$VA* \leq AV \Rightarrow (VA)^* VA = (VA)^* AV$ and $R(VA) \subseteq \mathrm{R}(AV)$

$\Rightarrow A^* V^* VA = A^* V^* AV$
$\Rightarrow A^* VVA = A^* VAV$
$\Rightarrow A^* A = A^* VAV$
$\Rightarrow I = A^* VAV$ [Since A is unitary]

$\Rightarrow VI = (VA^*V)AV \Rightarrow V = A^\theta AV$
$\Rightarrow V^2 = A^\theta AV^2$
$\Rightarrow I = AA^\theta$ [Since $V^2 = I$]

$\Rightarrow A^{-1} = A^\theta$

Therefore A is s-unitary.

Theorem 3.5 Let $AV \leq *VA$ .If A is unitary then A is s-unitary.

Proof: **:** Let $AV \leq *VA$.

$AV \leq *VA \Rightarrow AV(AV)^* = VA(AV)^*$ and

$R((AV)^*) \subseteq \mathrm{R}((VA)^*)$
$\Rightarrow AVV^* A^* = VAV^* A^* \Rightarrow AVVA^* = VAVA^*$
$\Rightarrow AIA^* = VAVA^*$ $\Rightarrow A^* A = VAVA^*$
$\Rightarrow I = VAVA^*$ $\Rightarrow IV = VA(VA^*V)$
$\Rightarrow V = VAA^\theta \Rightarrow V^2 = V^2 AA^\theta$
$\Rightarrow I = AA$
$\Rightarrow A^{-1} = A^\theta$

Therefore A is s-unitary.

Theorem 3.6 Let $VA \leq *AV$. If A is unitary then A is s-unitary.

Proof: Let $VA \leq *AV$.

$VA \leq *AV \Rightarrow VA(VA)^* = AV(VA)^*$ and

$R((VA)^*) \subseteq \mathrm{R}((AV)^*)$
$\Rightarrow VAA^* V^* = AVA^* V^*$
$\Rightarrow VIV = A(VA^*V) = AA^\theta$
$\Rightarrow I = AA^\theta$
$\Rightarrow A^{-1} = A^\theta$

Therefore A is s-unitary.

4.RELATION BETWEEN LEFT-STAR AND RIGHT -STAR PARTIAL ORDERING

Theorem 4.1 Let A and B be s-unitary matrices and if $R(A) = R(A^*), R(B) = R(B^*)$ then $A* \leq B \Rightarrow A^{-1} \leq *B^{-1}$

Proof:

$A* \leq B \Rightarrow A^* A = A^* B$ and $R(\mathrm{A}) \subseteq \mathrm{R}(\mathrm{B})$.

$\Rightarrow VA^* AV = VA^* BV$
$\Rightarrow VA^* VVAV = VA^* VVB\, V$ [Since $V^2 = I$]

$\Rightarrow (VA^*V)(VAV) = (VA^*V)(VBV)$
$\Rightarrow A^{-1}(VAV) = A^{-1}(VBV)$

Taking conjugate transpose of the above equation we get

$(VAV)^* (A^{-1})^* = (VBV)^* (A^{-1})^*$
$\Rightarrow (V^* A^* V^*)(A^{-1})^* = (V^* B^* V^*)(A^{-1})^*$
$\Rightarrow (VA^*V)(A^{-1})^* = (VB^*V)(A^{-1})^*$
$\Rightarrow (A^{-1})(A^{-1})^* = (B^{-1})(A^{-1})^* ---(1)$

Given $R(A) = R(A^*), R(B) = R(B^*)$

Therefore $R(A^*) \subseteq R(B^*) ---(2)$

From (1) and (2) we get

$A* \leq B \Rightarrow A^{-1} \leq *B^{-1}$

Theorem 4.2 Let A and B be s-unitary matrices and if $R(A) = R(A^*), R(B) = R(B^*)$ then $A \leq *B \Rightarrow A^{-1}* \leq B^{-1}$

Proof: $A \leq *B \Rightarrow AA^* = BA^*$ and $R(\mathrm{A}^*) \subseteq \mathrm{R}(\mathrm{B}^*)$

$\Rightarrow VAA^*V = VAB^*V$
$\Rightarrow VAVVA^*V = VAVVB^*V$ [since $V^2 = I$]

$\Rightarrow (VAV)(VA^*V) = (VAV)(VB^*V)$
$\Rightarrow (VAV)A^{-1} = (VAV)(B^{-1})$

Taking conjugate transpose of above equation we get

$\Rightarrow (A^{-1})^* (VAV)^* = (B^{-1})^* (VAV)^*$
$\Rightarrow (A^{-1})^* (V^* A^* V^*) = (V^* A^* V^*)(B^{-1})^*$
$\Rightarrow (A^{-1})^* (VA^*V) = (VA^*V)(B^{-1})^*$
$\Rightarrow (A^{-1})^* (A^{-1}) = (A^{-1})(B^{-1})^* ---(1)$

Given $R(A) = R(A^*), R(B) = R(B^*)$

Therefore $R(A) \subseteq R(B)$ ---(2)

From (1) and (2) we get $A \leq *B \Rightarrow A^{-1}* \leq B^{-1}$

Theorem 4.2 Let A and B s-unitary matrices .

$A* \leq B \quad B* \leq C \Rightarrow A* \leq C$

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Proof:

$A{*}\leq B \Rightarrow A^{*}A = A^{*}B$ and $R(\mathrm{A})\subseteq \mathrm{R(B)}$

$B{*}\leq C \Rightarrow B^{*}B = B^{*}C$ and $R(B)\subseteq R(\mathrm{C})$.

$B{*}\leq C \Rightarrow B^{*}B = B^{*}C$

$\Rightarrow B^{*}AA^{*}B = B^{*}C$ Since $AA^{*} = I$

$\Rightarrow B^{*}A(AA^{*}) = B^{*}C$

$\Rightarrow (B^{*})^{-1}B^{*}AA^{*}A = (B^{*})^{-1}B^{*}C$

$\Rightarrow AA^{*}A = C \Rightarrow A^{*}A = AC$

$\Rightarrow A^{*}A = A^{*}C$ ----(1)

Given $R(\mathrm{A})\subseteq \mathrm{R(B)}$ and $R(B)\subseteq R(\mathrm{C})$

Therefore $R(\mathrm{A})\subseteq R(B)$, $R(B)\subseteq R(\mathrm{C}) \Rightarrow R(\mathrm{A})\subseteq R(\mathrm{C})$ ---(2)

From (1) and (2) we get $A{*}\leq C$

$A{*}\leq B \quad B{*}\leq C \Rightarrow A{*}\leq C$

Theorem 4.3 If A be any matrix and B be hermitian matrices then

$A{*}\leq B \Rightarrow A^{*}\leq{*}B^{*}$

Proof:

$A{*}\leq B \Rightarrow A^{*}A = A^{*}B$ and $R(\mathrm{A})\subseteq \mathrm{R(B)}$

$A^{*}A = A^{*}B \Rightarrow A^{*}(A^{*})^{*} = (A^{*})^{*}B$

$\Rightarrow A^{*}(A^{*})^{*} = (A^{*})^{*}B^{*}$

Since is hermitian $B^{*} = B \Rightarrow A^{*}\leq{*}B^{*}$

and $R(A)\subseteq R(B) \Rightarrow R(A^{*})\subseteq R(B^{*})$

Therefore $A{*}\leq B \Rightarrow A^{*}\leq{*}B^{*}$

Theorem 4.5 If A be any matrix and B be hermitian matrices then $A\leq{*}B \Rightarrow A^{*}{*}\leq B^{*}$

Proof: Similar to previous one.

CONCLUSIONS

The above study has clearly indicated the relationship between left-star partial ordering and right-star partial ordering of s-unitary matrices. Also shown some results of left-star partial ordering and right-star partial ordering of s-unitary matrices.

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