# Construction of Partially balanced incomplete block designs based on Magic square of odd and doubly-even order 

## KEYWORDS

PBIB designs, magic square, block designs, treatments, replications

Davinder Kumar Garg
Department of Statistics, Punjabi University, Patiala-147002 (India)

Syed Aadil Farooq
Department of Statistics, Punjabi University, Patiala-147002 (India)


#### Abstract

In this paper, we have constructed two new series of partially balanced incomplete block (PBIB) designs by establishing a link between PBIB designs and magic square of odd order and doubly - even order. IIlustrations of construction of designs are also discussed in detail. Efficiencies of the newly constructed designs are also computed for the purpose of comparisons. It was found that some newly constructed designs are more efficient as compared to the existing designs and further these designs are useful from practical point of view.


## Introduction

In design of experiments the main problem is to test the effect of treatments by developing efficient designs under the given constraints with minimum error. For this, a number of methods of construction of partially balanced incomplete block designs are available in literature. One of the simplest methods of constructing partially balanced incomplete block (PBIB) designs is based on magic square. We study that Tomba [2012a] using basic latin squares for constructing a technique of Odd-order Magic Squares. Also, Tomba [2012b] used a Technique for constructing even - order Magic Squares. Recently, Tomba and Shibiraj [2013] introduced a technique for constructing doublyeven Magic Squares using basic latin squares.

A magic square of order $n$ is a square matrix or array of numbers consisting of $n$ rows and $n$ columns such that the sum of the elements of each row, each column, the main diagonal and main back diagonal, is the same number, called the magic constant (or magic sum, or line-sum). Generally, the entries in the magic squares are thought of as the natural numbers $1,2, \ldots$, where each number is used exactly once in each row and each column; such magic squares will be referred to here as normal magic squares. Sometimes they are also called classical magic squares. As in matrices, the main diagonal entries running from "northwest" to "southeast". Also in matrices the main back diagonal entries running from "northeast" to "southwest."

## 2. Definitions

2.1. Magic square of odd orders: These are the squares whose order is not divisible by 2. The general formula for this can be written mathematically in the order is $n=2 m$ +1 , where $m$ is some integer. For example magic squares of orders can be 3,5,7 and 9 etc..
2.1.1. Magic square of double-even orders: These are the squares whose order is even and is divisible by 2 and 4. The general formula for this can be written mathematically for such order is $n=4 m$, where $m$ is some integer. For example magic squares of double even orders can be 4, 8 and 12 etc.
3. Association scheme of new two and three associate class PBIB designs
In this section, association schemes of new two and three
associate class partially balanced Incomplete Block designs are defined as follows:
3.1 Association scheme of new two associate PBIB designs
Let there be $v=n^{2}$ treatments and $b=2 n$ blocks. In these $b=2 n$ blocks, every treatment repeats exactly $r=2$ times. Now for $v=\mathrm{n}^{2}$ treatments, we define a two associate class association scheme as follows: Let us consider any two blocks say P and Q which has exactly one treatment in common namely, say $\Theta$. All treatments which occurs with $\Theta$ are termed as first associate of each other i.e. $\lambda_{1}=1$ and $n_{1}=2(n-1)$, and all the remaining treatments which are not present in these blocks are second associate of each other i.e. $\lambda_{2}=0$ and $n_{2}=(n-1)^{2}$.

The parameters of two associate class association schemes are as follows:
$v=n^{2}, b=2 n, r=2, k=n, \lambda_{1}=1, \lambda_{2}=0, n_{1}=2(n-1), n_{2}$ $=(n-1)^{2}$

### 3.1.1 Association scheme of new three associate PBIB designs

This association scheme is almost similar to the rectangular association scheme. In this association scheme, we have v $=n^{2}$ treatments and $b=2 n$ blocks. In these $b=2 n$ blocks, every treatment repeats exactly $r=2$ times. Now we define a three associate class association scheme as follows: Let us take any two treatments say $\Theta$ and $\Phi$ between these blocks. These treatments are first associate of each other, if they occur in the same row, second associate of each other if they belong to the same column and are mutually third associates of each other if they neither occur in the same row nor same column of the array.

Here, $v=n^{2}, b=2 n, r=2, k=n, \lambda_{1}=1, \lambda_{2}=1, \lambda_{3}=0, n_{1}=$ $n-1, n_{2}=n-1, n_{3}=(n-1)^{2}$

## 4. Construction method of new two and three associate class PBIB designs:

In this section we introduce a new method of construction based on magic square of odd order and doubly even order. Also in this section, two illustrations of construction along with their parameters are also discussed in detail. The general formula for this order is $n=2 m$, where $m$ is some integer. This method consists of the following steps.
4.1.1 Construction method of constructing two associate PBIB designs using magic squares of odd order
Let us take $v=n^{2}$ treatments in $n$ rows and $n$ columns of $n \times n$ square arranged in a magic square format. In this method, firstly we write symbol 1 in the top center position of the first row. The numbers $2,3,4, \ldots, n^{2}$ are then placed in the remaining unoccupied positions of the $n \times$ $n$ square matrix according to the given pattern. Move one cell up and one cell to right to place treatment 2 . Since treatment 2 is off the grid, so move to the bottom of the grid, as through it wraps around. Place treatment 2 in unoccupied position of the bottom right hand corner, then moving diagonally upward (towards the "northeast") until we reach the right hand side. Then pick up on the left hand side on the next row up, entering the natural numbers through $n$, until reaching 1 again. The next number, which would be $n+1$, goes directly under $n$. Continue in the same pattern until whole of the unoccupied position in the $n \times n$ square is filled. If the top is reached, the next number is placed in the bottom square of the next column. By augmenting first $(n-1)^{\text {th }}$ columns to the right of the $n \times n$ matrix, then move diagonally running from top left to bottom right and from top right to bottom left forms blocks of the design with $b=2 n$, each of size $k=n$. Also, in these blocks, each treatment replicated once with $r=2$. Treating these $\mathrm{v}=\mathrm{n}^{2}$ as treatments and $\mathrm{b}=2 \mathrm{n}$ as blocks, we get a PBIB designs with the following parameters: $v$ $=n^{2}, b=2 n, r=2, k=n, \lambda_{1}=1, \lambda_{2}=0, n_{1}=2(n-1), n_{2}=$ $(n-1)^{2}$

### 4.1.2 Construction method of constructing two associate PBIB designs using magic squares of doubly-even order

In this construction method, we take treatment $v=n^{2}$ treatments arranged in ' $n$ ' rows and ' $n$ ' columns of the $n$ $\times \mathrm{n}$ square matrix, where n is double-even order. This method is similar to the method given by Durer [1514]. In this, we take First Square and fill the unoccupied positions of the square in the increasing order by the treatments $1,2,3 \ldots n^{2}$ by moving horizontally to the right in each row. Secondly, in the second square, fill the positions of the main diagonal by the treatments in the cells of the
main diagonal in the first square. Insert the cells of the third square by the treatments $1,2,3, \ldots, n^{2}$ by moving horizontally to the left in each row. Fill the positions of the treatment in the fourth square which are not lying on the main diagonals of the third square. In the fifth square, we combine the second and fourth square to get the required magic square, which is taken as base for constructing PBIB designs. In this magic square all rows and columns are considered as blocks with $b=2 n$, each of size $k=n$. Also in these blocks, each treatment replicated 'r' times. Treating this $v=n^{2}$ as treatments and $b=2 n$ as blocks, we get a PBIB designs with the following parameters:
$v=n^{2}, b=2 n, r=2, k=n, \lambda_{1}=1, \lambda_{2}=1, \lambda_{3}=0, n_{1}=n-1, n_{2}$ $=n-1, n_{3}=(n-1)^{2}$

## 5. List of PBIB designs with two and three associate classes

Here, we enlist some new designs for different values of parameters along with theirs efficiencies which are given in table 5.1.1 and 5.1.2

Table - 5.1.1
Table of two associate class PBIB designs ( $n>1$ ) for odd

| c | a |  |  |  | S |  |  |  | e |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\begin{aligned} & \dot{\circ} \\ & \stackrel{C}{C} \\ & \stackrel{-}{0} \\ & 0 \\ & 0 \end{aligned}$ | $\begin{aligned} & \mathrm{n} \\ & =\mathrm{v} \end{aligned}$ | b | r | k | $\lambda 1$ | $\lambda 2$ | $\mathrm{n}_{2}$ | $\mathrm{n}_{2}$ | $\mathrm{E}_{1}$ | $\mathrm{E}_{2}$ | E |
| 1 | 9 | 6 | 2 | 3 | 1 | 0 | 4 | 4 | 0.750 | 0.600 | 0.666 |
| 2 | 25 | 10 | 2 | 5 | 1 | 0 | 8 | 16 | 0.833 | 0.714 | 0.750 |
| 3 | 49 | 14 | 2 | 7 | 1 | 0 | 12 | 36 | 0.875 | 0.777 | 0.800 |
| 4 | 81 | 18 | 2 | 9 | 1 | 0 | 16 | 64 | 0.900 | 0.818 | 0.833 |
| 5 | 121 | 22 | 2 | 11 | 1 | 0 | 20 | 100 | 0.984 | 0.909 | 0.920 |

Table-5.1.2
Table of three associate class PBIB designs ( $n>2$ ) for doubly even cases

| $\begin{aligned} & \dot{C} \\ & \stackrel{\circ}{n} \\ & 0 \\ & 0 \\ & 0 \\ & \hline Z \end{aligned}$ | $\mathrm{s}=\mathrm{v}$ | b | r | k | $\lambda 1$ | $\lambda 2$ | $\lambda 3$ | $\mathrm{n}_{1}$ | $\mathrm{n}_{2}$ | $\mathrm{n}_{3}$ | $\mathrm{E}_{1}$ | $\mathrm{E}_{2}$ | $\mathrm{E}_{3}$ | E |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 16 | 8 | 2 | 4 | 1 | 1 | 0 | 3 | 3 | 9 | 0.800 | 0.800 | 0.666 | 0.717 |
| 2 | 64 | 16 | 2 | 8 | 1 | 1 | 0 | 7 | 7 | 49 | 0.896 | 0.882 | 0.795 | 0.814 |
| 3 | 144 | 24 | 2 | 12 | 1 | 1 | 0 | 11 | 11 | 121 | 0.923 | 0.923 | 0.857 | 0.866 |
| 4 | 256 | 32 | 2 | 16 | 1 | 1 | 0 | 15 | 15 | 225 | 0.941 | 0.941 | 0.888 | 0.894 |
| 5 | 400 | 40 | 2 | 20 | 1 | 1 | 0 | 19 | 19 | 361 | 0.952 | 0.952 | 0.909 | 0.913 |


#### Abstract

6. Conclusion

In this paper, we have constructed two new series of PBIB designs by establishing a link between PBIB designs and magic square of odd order and double - even order. In this paper, we have kept number of replications fixed and allowed block size to vary as numbers of treatments vary. As number of treatments as well as blocks size increases rapidly as compared to number of replications, the efficiencies of new designs also increase, which is a significant feature. We have listed the parameters along with their efficiencies in Table - 5.1.1 and Table - 5.1.2 respectively. From Table 5.1.1, we see that design number 5 is new in the sense that these designs are not listed in Clatworthy [1973]. Some designs coming from both series are quite useful in particular treatment combinations due to better efficiencies as compared to the other two associate class PBIB designs and some already existing higher associate class PBIB designs.


REFERENCE Clatworthy, W.H. (1973). Tables of two associate class Partially Balanced Incomplete Block Designs. NBS, Applied Mathematics Ser. No. 63. Dey, A. (2010). Incomplete Block Designs. Hindustan Book Agency. New Delhi. | Durer, A. (1514). Melancholia I, The British Museum, Burton 1989, Gellert et al. 1989. (engraving). | John, J.A and Williams, E.R (1995). Cyclic and computer generated designs second edition.Springer-Science+Business Media, B.V | Tomba, I. (2012a). A Technique for constructing Odd-order Magic Squares using Basic Latin Squares. International Journal of scientific and research Publications. 2(5), 1-5. | Tomba, I. (2012b). A Technique for constructing Even - order Magic Squares using Basic Latin Squares. International Journal of scientific and research Publications. 2(7),1-10. | Tomba, I. and Shibiraj, N. (2013). Improved Technique for Constructing Doubly-even Magic Squares using Basic Latin Squares. International Journal of Scientific and Research Publications. 3(6), 1-5. |

