A Comparison Between the Differential Transform Method and Homotopy Perturbation Method for a System of Non Linear Chemistry Problems

KEYWORDS: Homotopy perturbation method, Differential Transform Method, Inverse Differential Transform, system of non linear differential equation, numerical method, Chemistry problem.

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ABSTRACT *In this paper we implement relatively new analytical techniques, the Homotopy perturbation method and Differential transform method for solving a system of nonlinear algebraic differential equations. This paper will present a numerical comparison between the methods for solving system of nonlinear algebraic differential equations. Illustrative examples are given to demonstrate the validity and applicability of both the techniques.*

1. Introduction

Most scientific problems and phenomena occur nonlinearly. Except for a limited number of these problems, most of them do not have precise analytical solution; thus we have to use various approximate analytical methods. In the recent years, several numerical and analytical methods are being developed for solving non linear problems [Elçin Yusufoglu. (2009); Ercan Celik.et.al. (2003) ;(He, 2000a; 2003b; 2009c); Wazwaz, (2007)].

The subject of this present paper is to apply the differential transform method for solving a system of non linear algebraic differential equations and carry out the comparison with the Homotopy perturbation method. For this purpose we consider the system of non linear algebraic differential equations is of the form

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \ldots\ldots\ldots x_n), \qquad x_i(t) = a_i \quad \forall\, i = 1,2,3,\ldots\ldots n \tag{1}$$

Where a_i is a constant vector and f_i is the solution vector..

This equation has been examined by several numerical methods such as Adomian Decomposition Method , Variational Iteration Method [Ganji et al (2007)] and Homotopy Analysis Method [Matinfar et al (2014)].

The basic ideas and current developments of differential transform method to get convergent series solution of non linear differential equation which recently attracts interests of more researchers. The concept of the differential tTransform Method (DTM) was first introduced by Zhou [Zhou, (1986)] and has been used to solve both linear and non linear initial value problems in electric circuit analysis. It is a semi numerical analytic technique that formalizes Taylor series in a totally different manner [Vedat Suat Ertürk and Shaher Momani, (2008); Tari and Shahmorad,(2011)]. The main advantage of DTM is that it can be directly to problems without requiring linearization, discretization or perturbation. It has been used to solve effectively and accurately a large class of nonlinear problems with approximations.

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The differential transform method has been successfully applied to solve system of differential equations in [Mossa Al-sawalha and Noorani, (2009)],differential algebraic equations[Ayaz,(2004)],difference equations[Arikoglu and Ozkol, 2006 a],differential difference equations[Arikoglu and. Ozkol, 2006b],partial differential equations [Borhanifar and bazari,(2011); Soltanalizadeh, (2011)], variational problems [Ramesh Rao, (2013)] and Burgers equations[Abazari and Borhanifar, (2010); Reza Abazari, and Borhanifar, (2010)].

The organization of the rest of this paper is as follows; in section 2, we implement the Differential Transform Method is applied to the system of non linear differential equations. In section 3, we extend the application of the Homotopy perturbation method to construct analytical approximate solution to the system of non linear differential equations. To present a clear overview of both the methods, numerical examples are given in the section 4. A conclusion is presented in section 5.

2. Basic idea of Differential Transform Method

The differential transform of the function $y(x)$ is defined as

$$F(k)=\frac{1}{k!}\left[\frac{d^k}{dx^k}y(x)\right]_{x=x_0} \tag{2}$$

Where $y(x)$ the original is function and $F(k)$ is the transformed function. Differential inverse transform of $F(k)$, is defined as $y(x)=\sum_{k=0}^{\infty}F(k)\,(x-x_0)^k$ (3)

Combining equations (2) and (3), we arrive at:

$$y(x)=\sum_{k=0}^{\infty}(x-x_0)^k\,\frac{1}{k!}\left[\frac{d^k}{dx^k}y(x)\right]_{x=x_0}$$

In real applications, the function $y(x)$ is expressed by a finite series and Eqn. (3) can be written as

$$y(x)=\sum_{k=0}^{n}F(k)\,(x-x_0)^k \tag{4}$$

The following theorems that can be deduced from Eqns. (2) and (3).For more details see the mentioned references.

Theorem 1. If $y(x)=u(x)\pm v(x)$ then $F(k)=U(k)\pm V(k)$

Theorem 2. If $y(x)=c\,u(x),$ then $F(k)=c\;U(k),$ where c is a constant.

Theorem 3. If $y(x)=\frac{d^n}{dx^n}[u(x)],$ then $F(k)=\frac{(k+n)!}{k!}U(k+n).$

Theorem 4. If $y(x)=u(x)\,v(x)$ then $F(k)=\sum_{k_1=0}^{k} U(k_1)\,V(k-k_1)$.

Theorem 5. If $y(x)=x^n$ then $F(k)=\delta(k-n)=\begin{cases}1 & if\ k=n\\ 0 & if\ k\neq n\end{cases}$

Theorem 6. If $y(x)=u_1(x)\,u_2(x)u_3(x).........u_{n-1}(x)\;u_n(x)$ then

$$F(k)=\sum_{k_{n-1}=0}^{k_n}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_1=0}^{k_2}U_1(k_1)U_2(k_2-k_1)\cdots U_n(k-k_{n-1})$$

3. Basic idea of Homotopy perturbation method

The basic idea of this method is the following. We consider the non linear differential equation

$$A(u)-f(r)=0, \qquad r\in\Omega \tag{5}$$

With boundary conditions

$$B\left(u,\frac{\partial u}{\partial n}\right)=0 \qquad r\in\Gamma \tag{6}$$

Where A is the general differential operator, B is the boundary operator; $f(r)$ is a known analytic function. Γ is the boundary of the domain Ω.

The operator A can be generally divided in to two parts L and N, where L is linear, whereas N is non linear. Therefore, Eq. (5) can be rewritten as follows:

$$L(u)+N(u)-f(r)=0, \qquad r\in\Omega \tag{7}$$

Using the homotopy technique, we construct the homotopy as: $v(r,p):\Omega\ X[0,1]\rightarrow\Re$ which satisfies:

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[A(v)-f(r)]=0, \qquad r\in\Omega \tag{8}$$

Where $p\in[0,1]$ is an embedding parameter and u_0 is the first approximation that satisfy the boundary conditions. We can write this homotopy equation as follows:

$$H(v,p)=L(v)-L(u_0)+p\,L(u_0)+p[N(v)-f(r)]=0, \qquad r\in\Omega \tag{9}$$

Obviously, from eqns. (8) and (9), we have:

$$H(v,0)=L(u)-L(u_0)=0 \tag{10}$$

$$H(v,1)=L(u)+N(u)-f(r)=0 \tag{11}$$

The embedding parameter p monotonically changes from zero to unity as the trivial problem $L(v)-L(u_0)$ is continuously deformed to the original problem $A(u)-f$. Due to the fact that $p\in[0,1]$ can be considered as a small parameter, hence we consider the solution of Eqn.(8) as a power series in p as the following:

$$v=v_0+pv_1+p^2v_2+p^3v_3+...................... \tag{12}$$

When $p\rightarrow 1$, Eqn. (8) correspond to the Eqn.(4) becomes the approximate solution of (3),ie.,

$$u(x)=\lim_{p\to 1} v = v_0 + v_1 + v_2 + \dots\dots\dots \tag{13}$$

The series (13) is convergent for most of the cases, and the rate of convergence depends on $A(v)$,(see [He, (1999)]).

4. Test Examples

4.1 Example 1

As the first examples consider the following problem consisting of three non linear differential equations which describe the kinetics of auto catalytic reaction [Robertson, (1966)]:

$$\begin{aligned}
\frac{dx}{dt} &= -c_1 x + c_2 yz \\
\frac{dy}{dt} &= c_3 x - c_4 yz - c_5 y^2 \\
\frac{dz}{dt} &= c_6 y^2
\end{aligned} \tag{14}$$

Where c_1, c_2, c_3, c_4, c_5 and c_6 are constant parameters

($c_1 = 0.04, c_2 = 0.01, c_3 = 400, c_4 = 100, c_5 = 30000, c_6 = 30$). The initial conditions are given by

$$x(0)=1,\ y(0)=0 \text{ and } z(0)=0 \tag{15}$$

DTM approach

Applying the differential transform theorems to Eq. (14), we obtain the following recurrence relation:

$$\begin{aligned}
(k+1)F_1(k+1) &= -c_1 F_1(k) + c_2 \sum_{k_1=0}^{k} F_2(k_1)\, F_3(k-k_1) \\
(k+1)F_2(k+1) &= c_3 F_1(k) - c_4 \sum_{k_1=0}^{k} F_2(k_1)\, F_3(k-k_1) - c_5 \sum_{k_1=0}^{k} F_2(k_1)\, F_2(k-k_1) \\
(k+1)F_3(k+1) &= c_6 \sum_{k_1=0}^{k} F_2(k_1)\, F_2(k-k_1)
\end{aligned} \tag{16}$$

Using the Eqn. (2) and (15), the initial conditions at $x_0 = 0$ can be transformed as follows:

$$F_1(0)=1,\ F_2(0)=0,\ F_3(0)=0 \tag{17}$$

By using the recurrence relation in Eq.(16) and the transformed initial conditions in Eq.(17), the following series solution up to $O(t^6)$ is obtained:

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$$x(t)=1-c_1t+\frac{c_1^{\,2}}{2}t^2-\frac{c_1^{\,3}}{6}t^3+\frac{c_1^{\,4}}{24}t^4+\left(\frac{-c_1^{\,5}}{120}+\frac{c_2c_3^{\,3}c_6}{15}\right)t^5+O\left(t^6\right)$$

$$y(t)=c_3t-\frac{c_1c_3}{2}t^2+\frac{c_1^{\,2}c_3}{6}t^3+\left(\frac{c_1c_3^{\,2}c_5}{4}-\frac{c_1^{\,3}c_3}{24}\right)t^4+$$
$$\left(\frac{c_1^{\,4}c_3}{120}-\frac{c_3^{\,3}c_4c_6}{15}-\frac{7}{60}c_1^{\,2}c_3^{\,2}c_5\right)t^5+O\left(t^6\right)$$

$$z(t)=\frac{c_3^{\,2}c_6}{3}t^3-\frac{c_1c_3^{\,2}c_6}{4}t^4+\frac{7}{60}c_1^{\,2}c_3^{\,2}c_6\,t^5+O\left(t^6\right)$$

(18)

HPM approach

To solve the system (14) by the HPM, we construct the homotopy as

$$\frac{dx}{dt}-\frac{dx_0}{dt}=p\left(-c_1x+c_2yz-\frac{dx_0}{dt}\right)\qquad x_0=1,\ y_0=0=z_0$$
$$\frac{dy}{dt}-\frac{dy_0}{dt}=p\left(c_3x-c_4yz-c_5y^2-\frac{dy_0}{dt}\right)\tag{19}$$
$$\frac{dz}{dt}-\frac{dz_0}{dt}=p\left(c_6y^2-\frac{dz_0}{dt}\right)$$

Substituting (12) in to (19) and equating the terms with identical powers of p , we have

$$p^{(0)}:\begin{cases}\dfrac{dx_0}{dt}-\dfrac{dx_0}{dt}=0\\[2mm]\dfrac{dy_0}{dt}-\dfrac{dy_0}{dt}=0\\[2mm]\dfrac{dz_0}{dt}-\dfrac{dz_0}{dt}=0\end{cases}$$

(20)

$$p^{(1)}:\begin{cases}\dfrac{dx_1}{dt}=-c_1x_0+c_2y_0z_0-\dfrac{dx_0}{dt}\\[2mm]\dfrac{dy_1}{dt}=c_3x_0-c_4y_0z_0-c_5y_0^{\,2}-\dfrac{dy_0}{dt}\\[2mm]\dfrac{dz_1}{dt}=c_6y_0^{\,2}-\dfrac{dz_0}{dt}\end{cases}$$

(21)

$$p^{(2)}:\begin{cases}\dfrac{dx_2}{dt}=-c_1x_1+c_2y_1z_1\\[2mm]\dfrac{dy_2}{dt}=c_3x_1-c_4y_1z_1-c_5y_1^{\,2}\\[2mm]\dfrac{dz_2}{dt}=c_6y_1^{\,2}\end{cases}$$

(22)

$$p^{(2)}:\begin{cases}\dfrac{dx_3}{dt}=-c_1x_2+c_2y_2z_2\\ \dfrac{dy_3}{dt}=c_3x_2-c_4y_2z_2-c_5{y_2}^2\\ \dfrac{dz_3}{dt}=c_6{y_2}^2\end{cases} \tag{23}$$

Solving the system accordingly and using (13), we obtain

$$x(t)=1-c_1t+\frac{{c_1}^2}{2}t^2-\frac{{c_1}^3}{6}t^3+\frac{{c_1}^4}{24}t^4+\frac{c_2{c_3}^3c_6}{15}t^5+.......$$

$$y(t)=c_3t-\frac{c_1c_3}{2}t^2+\left(\frac{{c_1}^2c_3}{6}t^3-\frac{c_5{c_3}^2}{3}\right)t^3+\left(\frac{c_1{c_3}^2c_5}{4}-\frac{{c_1}^3c_3}{24}\right)t^4-\frac{{c_3}^3c_4c_6}{15}t^5+...... \tag{24}$$

$$z(t)=\frac{{c_3}^2c_6}{2}t^2-\frac{c_1{c_3}^2c_6}{4}t^4+..........$$

4.2 Example 2

The second example is a system representing a nonlinear reaction, which was taken from Hull et al [1972]:

$$\frac{dx}{dt}=-x$$
$$\frac{dy}{dt}=x-y^2$$
$$\frac{dz}{dt}=y^2 \tag{25}$$

with initial conditions

$$x(0)=1,\ y(0)=0 \text{ and } z(0)=0 \tag{26}$$

DTM approach

Applying the differential transform theorems to Eq. (25), we obtain the following recurrence relation:

$$(k+1)F_1(k+1)=-F_1(k)$$
$$(k+1)F_2(k+1)=F_1(k)-\sum_{k_1=0}^{k}F_2(k_1)\,F_2(k-k_1)$$
$$(k+1)F_3(k+1)=\sum_{k_1=0}^{k}F_2(k_1)\,F_2(k-k_1) \tag{27}$$

Using the Eqn. (2) and (26), the initial conditions at $x_0=0$ can be transformed as follows:

$$F_1(0)=1,\ F_2(0)=0,\ F_3(0)=0 \tag{28}$$

By using the recurrence relation in Eq. (27) and the transformed initial conditions in Eq. (28), the following series solution up to O (t^7) is obtained:

$$x(t)=1-t+\frac{1}{2}t^2-\frac{1}{3}t^3+\frac{1}{24}t^4-\frac{1}{120}t^5+\frac{1}{720}t^6+O\left(t^7\right)$$
$$y(t)=t-\frac{1}{2}t^2+\frac{1}{3}t^3+\frac{7}{24}t^4-\frac{1}{8}t^5+\frac{31}{720}t^6+O\left(t^7\right) \tag{29}$$
$$z(t)=\frac{1}{3}t^3-\frac{1}{3}t^4+\frac{2}{15}t^5-\frac{2}{45}t^6+O\left(t^7\right)$$

HPM approach

Now we apply HAM to Eqn. (25), we construct homotopy as

$$\frac{dx}{dt}-\frac{dx_0}{dt}=p\left(-x-\frac{dx_0}{dt}\right) \qquad x_0=1,\ y_0=0=z_0$$
$$\frac{dy}{dt}-\frac{dy_0}{dt}=p\left(x-y^2-\frac{dy_0}{dt}\right) \tag{30}$$
$$\frac{dz}{dt}-\frac{dz_0}{dt}=p\left(y^2-\frac{dz_0}{dt}\right)$$

Substituting (12) in to (30) and equating the terms with identical powers of p , we have

$$p^{(0)}:\begin{cases}\dfrac{dx_0}{dt}-\dfrac{dx_0}{dt}=0\\[2mm]\dfrac{dy_0}{dt}-\dfrac{dy_0}{dt}=0\\[2mm]\dfrac{dz_0}{dt}-\dfrac{dz_0}{dt}=0\end{cases} \tag{31}$$

$$p^{(1)}:\begin{cases}\dfrac{dx_1}{dt}=-x_0-\dfrac{dx_0}{dt}\\[2mm]\dfrac{dy_1}{dt}=x_0-{y_0}^2-\dfrac{dy_0}{dt}\\[2mm]\dfrac{dz_1}{dt}={y_0}^2-\dfrac{dz_0}{dt}\end{cases} \tag{32}$$

$$p^{(2)}:\begin{cases}\dfrac{dx_2}{dt}=-x_1\\[2mm]\dfrac{dy_2}{dt}=x_1-{y_1}^2\\[2mm]\dfrac{dz_2}{dt}={y_1}^2\end{cases} \tag{33}$$

$$p^{(3)}:\begin{cases}\dfrac{dx_3}{dt}=-x_2\\[2mm]\dfrac{dy_3}{dt}=x_2-{y_2}^2\\[2mm]\dfrac{dz_3}{dt}={y_2}^2\end{cases}\tag{34}$$

Solving the system accordingly and using (13), we obtain

$$\begin{aligned}
(t)&=1-t+\frac{t^2}{2}-\frac{t^3}{6}+\frac{t^4}{24}-\frac{t^5}{120}+\frac{t^6}{720}-.....\\
(t)&=t-\frac{t^2}{2}-\frac{t^3}{6}+\frac{5}{24}t^4+\frac{t^5}{40}-\frac{67}{720}t^6+........\\
(t)&=\frac{t^3}{3}-\frac{t^4}{4}-\frac{t^5}{60}+\frac{7}{72}t^6+..........
\end{aligned}\tag{35}$$

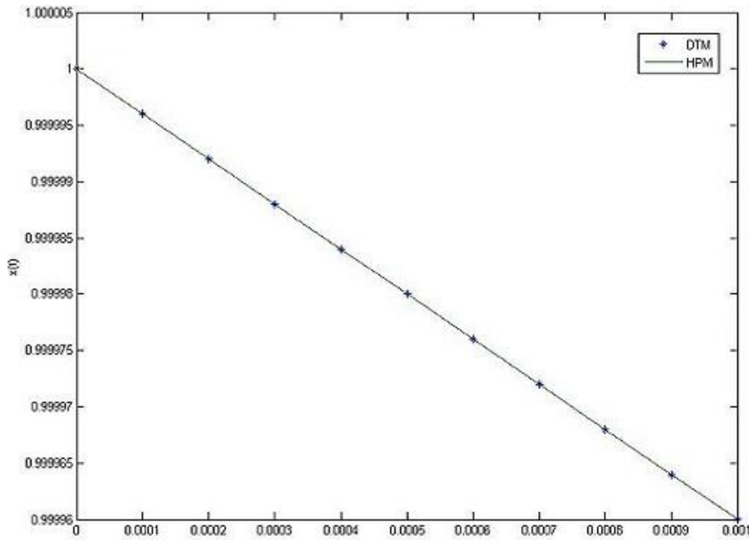

Fig. 1. The comparison of the results of x(t) via DTM and HPM for example 1.

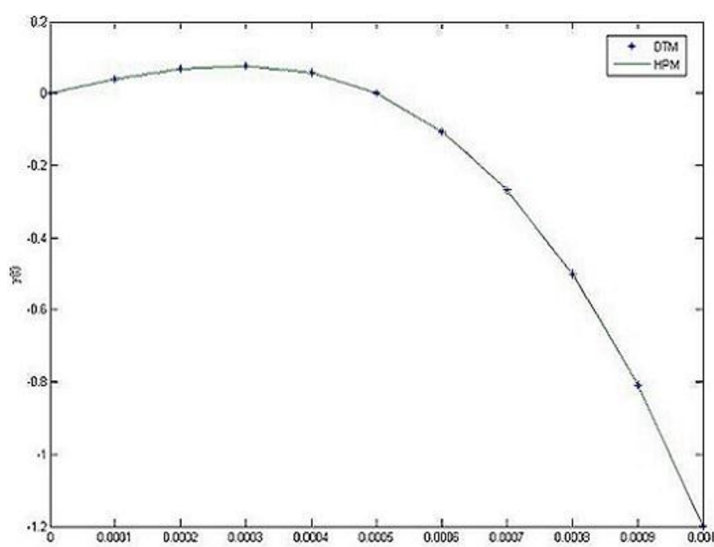

Fig. 2. The comparison of the results of y(t) via DTM and HPM for example 1

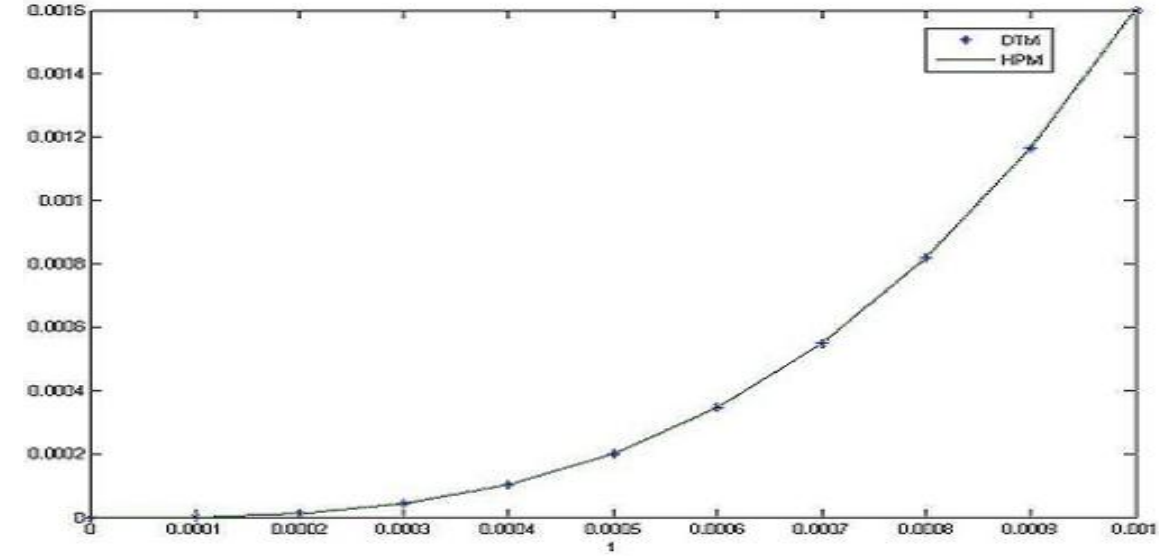

Fig. 3. The comparison of the results of z(t) via DTM and HPM for example 1

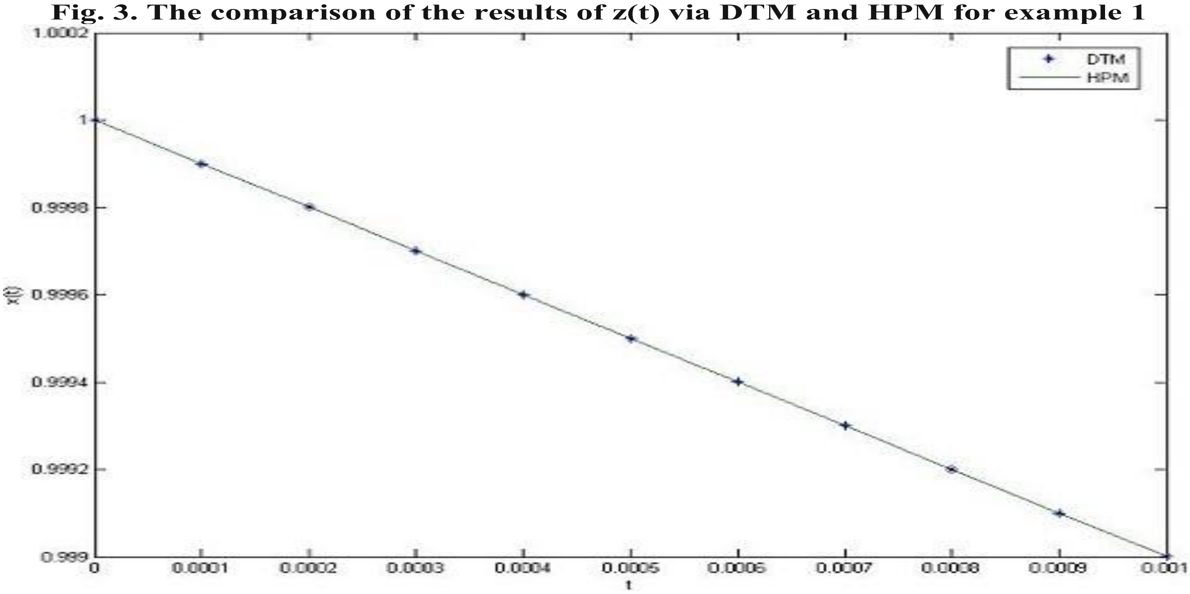

Fig. 4. The comparison of the results of x(t) via DTM and HPM for example 2.

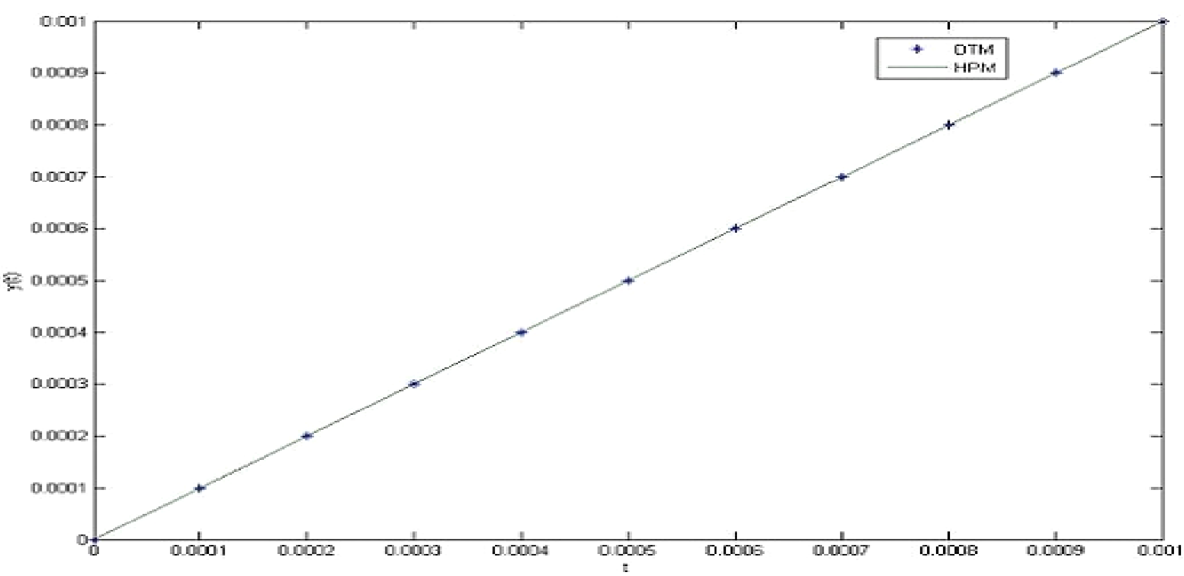

Fig. 5. The comparison of the results of y(t) via DTM and HPM for example 2.

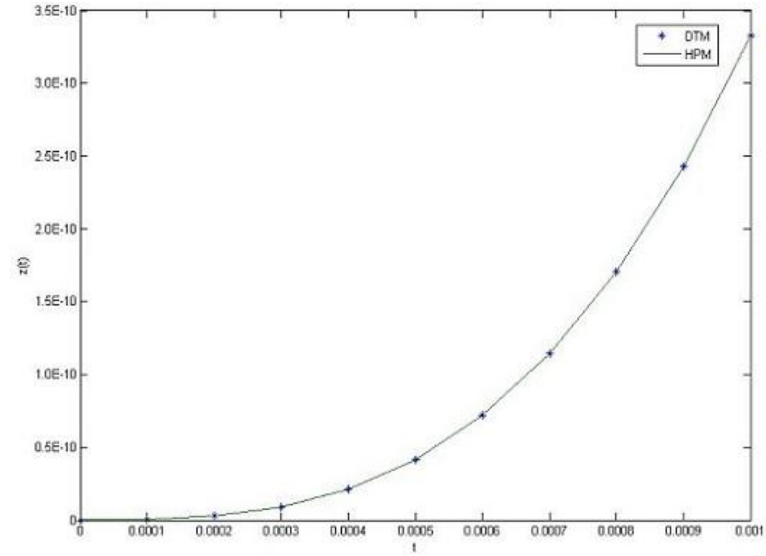

Fig. 6. The comparison of the results of z(t) via DTM and HPM for example 2.

Conclusions

In this paper, we have shown that the DTM and HPM can be used successfully for solving the system of non linear differential equations. Both of methods show that the results are in excellent agreement with together and the obtained solutions are shown graphically. In our work, we use the Matlab package to calculate the functions obtained from the DTM and HPM methods.

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