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On Integrability of Hamiltonian Systems

KEYWORDS

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ABSTRACT *Separation of variables is one of the methods for integrability of Hamil tonian systems . Other several methods are known. In this paper we treat the problem of integrability of Hamiltonian systems geometrically. The set up that we use is the Cartan method of moving frame. The killing tensor is the major entity that we use to determine the separation variables.*

1. Introduction

The problem of the itegrability of Hamiltonian system is a long standing problem. Several trials has been achieved to approach a complete solution. In fact there has been roughly three main approaches. The first approach is the classical approach where one seeks first integrals of the Hamiltonian system and then the solution is written via these integrals. A first integral F satisfies:

$\frac{\partial F}{\partial t} + \{F, H\} = 0$, Where $\{,\}$ is the Poisson bracket and H is the Hamiltonian function . In this approach one uses Calculus as an analytical tool. However one can also utilize Lie bracket instead of Poisson bracket. The Lie bracket is considered as a geometrical approach, where we involve vector fields, called Hamiltonian vector fields, corresponding to Hamiltonian functions. The next stage in the development of integrability of Hamiltonian system is due to Eisenhart and Cartan. Eiserhart used the frame field and Cartan used the coframe field and thus exterior Calculus is to be the geometrical tool for a free coordinates description of Hamiltonian system and Hamilton's equation. In this late approach we prove existance and uniqueness of solutions of the system , which is the problem of integrability . Of particular interest to us as a technique to solve Hamiltonian system is the method of separation of variables . The key idea behind this method is to seek a k-set of special coordinates : $q = (q^1, \dots, q^k)$ in which corresponding Hamilton- Jacobi partial differential equation admits a complete integral of the form

$$w(q, C) = w_1(q^1, C) + \cdots + w_n(q^n, C).$$

This method of separability has been considered by several mathematician such as Dall' Acqua, Eisenhart, Levi-Civita , Riai , Stackel and others. In this paper we develop the method of separability and use it in some cases.

Volume : 6 | Issue : 5 | May 2016 | ISSN - 2249-555X | IF : 3.919 | IC Value : 74.50

2. Preliminaries

1. The first and second fundament forms on a surface

Let $S \subset R^3$ be a surface and let the dot product of R^3 be given by $\langle .,. \rangle$. let a local chart fors be given by the map $X: V \to S$ where $V \subset R^2$ is an open set in a neighbourhood of a point we choose a local orthonormal frame, smooth vector fields $\{e_1, e_2, e_3\}$ such that

$$\langle e_i, e_j \rangle = \delta_{ij} \ , \qquad i, j = 1,2,3 \tag{1}$$

We choose the frame a dappled in such a way that e_3 is the unit normal vector and e_1 and e_2 span the tangent space T_pS. The corresponding coframe field of one forms $\{w^i\}$ is defined by the differential

$$dX = w^1 e_1 + w^2 e_2 \tag{2}$$

In local coordinated $(u^1, u^2) \in U$ the one forms are linear functional of the form

$$w(.) = p_1(u^1, u^2)\mathrm{du}^1 + p_2(u^1, u^2)\mathrm{du}^2 \tag{3}$$

Where p is a smooth function in U, du^i are the differential for the coordinate functions $u^i: U \to R$ which for a basis for the linear functional on the vector space $T_{(u^1,u^2)}U$.

The vectors in local coordinates have the expression

$$V = v^1(u^1, u^2)\frac{\partial}{\partial \mathrm{u}^1} + v^2(u^1, u^2)\frac{\partial}{\partial \mathrm{u}^2}$$

The one form (3) acts by:

$$w(v) = v^1(u^1, u^2)\mathrm{p}_1(u^1, u^2) + v^2(u^1, u^2)\mathrm{p}_2(u^1, u^2)$$

The usual identification between $X: U \to X(U)$ or $dX: T(u^1, u^2)\mathrm{U} \to \mathrm{T_pS}$, the one forms can be interpreted as linear functional on $\mathrm{T_pS}$ as well. For example if we choose vector fields E_i in U such that $dX(E_i) = e_i$ then we may set $\overline{w}(e_i) = w(E_i)$.

In particular $\overline{w}^i(e_j) = \delta^i_j$. It also means that metric takes the form

Volume : 6 | Issue : 5 | May 2016 | ISSN - 2249-555X | IF : 3.919 | IC Value : 74.50

$ds^2 = (w^1)^2 + (w^2)^2$ then $w^i$is a coframe and the vector fields e_i determined by duality $\overline{w}^i(e_j) = \delta^i_j$ are corresponding orthonormal frame.

Two one forms may be multiplied (wedged) to give a two-form, which is asxer symmetric bilinear form on the tangent space. For example if θ and ω are one forms then for vector field X, Y we have the form

$$(\theta \wedge \omega)(X, Y) = \theta(X)\omega(Y) - \theta(Y)\omega(X)$$

In local coordinates this gives

$$p_1 du^1 + p_2 du^2) \wedge (q_1 du^1 + q_2 du^2) = (p_1 q_2 - p_2 q_1)\, du^1 \wedge du^2$$

Because three vectors are dependent there are no skew symmetric three forms in R^2 and the most general two forms is

$$\beta = A(u^1, u^2) du^1 \wedge du^2.$$

When evaluated on the vectors

$$V = v^1(u^1, u^2)\frac{\partial}{\partial \mathrm{u}^1} + v^2(u^1, u^2)\frac{\partial}{\partial \mathrm{u}^2},$$

$$Z = z^1(u^1, u^2)\frac{\partial}{\partial \mathrm{u}^1} + z^2(u^1, u^2)\frac{\partial}{\partial \mathrm{u}^2}$$

The two forms give

$$\beta(V, Z) = A(u^1, u^2)(v^1(u^1, u^2)z^2(u^1, u^2) - v^2(u^1, u^2)z^1(u^1, u^2$$

The metric, has the expression form

$$dS^2 = \langle dX, dX \rangle = (w^1)^2 + (w^2)^2$$

Which is called the first fundamental form.

The Weingarten equations express the rotation of frame when moved along the surfaces

$$de_i = \sum w^i_j\, e_i \qquad (4)$$

In local coordinates the second fundamental form is given using (2), (4)

Volume : 6 | Issue : 5 | May 2016 | ISSN - 2249-555X | IF : 3.919 | IC Value : 74.50

$$\begin{aligned} \mathrm{II}(.,.) &= (de_3, X) \\ &= (w_3^1 e_1 + w_3^2 e_2, w^1 e_1 + w^2 e_2 \\ &= -w_3^1 \otimes w^1 - w_3^2 \otimes w^2 \\ &= w_1^3 \otimes w^1 - w_2^3 \otimes w^2 \end{aligned} \tag{5}$$

We may express the connection forms using the basis

$$\begin{aligned} w_1^2 &= h_{11} w^1 + h_{12} w^2 \\ w_2^3 &= h_{21} w^1 + h_{22} w^2 \end{aligned} \tag{6}$$

Thus inserting into (5):

$$(.,.) = \sum h_{ij} w^i \otimes w^j$$

In particular, if one searches through all unit tangent

$$V_\emptyset \coloneqq \cos(\emptyset)\, e_1 + \sin(\emptyset)\, e_2$$

For which $(V_\emptyset, V_\emptyset)$ is maximum and minimum, one finds that the extreme occur as eigenvectors of h_{ij} and the principal curvatures k_i are the corresponding eigenvalues. The GauB and mean curvatures are

2.2 Covariant differentiation:

Covariant differentiation of a vector field y in direction of another vector field $V = \sum v^i \frac{\partial}{\partial u^i}$ on U is a vector field denoted $\nabla_v y$. It is determined by orthogonal projection the tangent space $\nabla_v y \coloneqq proj\,(d(v))$. Hence in the local frame

$$\nabla_v y \coloneqq proj\,\big(d(v)\big) = \sum w_j^i(v)\, e_i$$

Covariant differentiation extends to all smooth vector fields, v, w on U and y, z on S and smooth functions $\emptyset, \varphi$ by the formulas:

i) $\nabla \emptyset v + \varphi w z = \emptyset \nabla_v z + \varphi \nabla_w z$ (linearily).
ii) $\nabla_v(\emptyset y + \varphi z) = x(\emptyset) y + \emptyset \nabla_v y + x(\varphi) z + \varphi \nabla_v z$ (lebinilz).
iii) $v\langle y, z\rangle = \langle \nabla_v y, z\rangle + \langle y, \nabla_v z\rangle$ (metric compatibility).

With these formulas one can deduce $\nabla_v(\sum y^i e_i)$. As

$$du^1 \wedge du^2 = 0$$

2.3 The structure equations:

differentiating (2) and (4)we get

$$0 = d^2X = d\left(\sum w^i e_i\right) = \sum dw^i e_i - \sum w^i \wedge de_i$$

$$= \sum dw^i e_i + \sum w^i \wedge w_i^1 e_i$$

$$0 = d^2 e_j = d\left(\sum w_j^k e_k\right) = \sum dw_j^k e_k - \sum w_j^k de_k$$

$$= \sum dw_j^k e_k - \sum w_j^k \wedge w_k^l e_l$$

Now collect coefficients for the basis vectors $e_j$and e_i

$$0 = dw^j - \sum w^i \wedge w_j^i,$$

$$0 = dw_j^l - \sum w_j^k \wedge w_k^l \tag{7}$$

These are called the first and second structure equations. By taking also the e_3 coefficient of d^2X,

$$0 = \sum w^i \wedge w_j^3$$

So that by (1.14) and (1.15)

$$0 = \sum w^i \wedge (h_{ij} w^j) = (h_{12} - h_{21}) w^1 \wedge w^2$$

It follows that $h_{ij} = h_{ji}$ is a symmetric matrix. In the test, we saw this when we proved the shape operator. de_3 was self adjoint.

The second structure equation enables us to compute the Gauss curvature form the connection matrix indeed by (1.14):

$$dw_1^2 = \sum w_1^i \wedge w_i^2 = w_1^3 \wedge w_3^2 = w_1^3 \wedge w_2^3$$

$$= -(h_{11} w^1 + h_{12} w^2) \wedge (h_{21} w^1 + h_{22} w^2)$$

Volume : 6 | Issue : 5 | May 2016 | ISSN - 2249-555X | IF : 3.919 | IC Value : 74.50

$$= -\left(h_{11}h_{22} - {h_{12}}^2\right) w^1 \wedge w^2 = -k\, w^1 \wedge w^2 \qquad (8)$$

The remarkable thing is that the conditions (1.12) and (1.15)

$$dw^i = \sum w^j \wedge w^i_j$$

$$w^j_i + w^i_j = 0 \qquad (9)$$

Determine w^2_1 uniquely. Since w^i is known once the metric is known by (1.4) this says that w^2_1 and thus k (an be determine form the metric alone).

2.4 Intrinsic Geometry:

Computation of curvature from the metric.

Let us compute the curvature of a metric in orthogonal coordinates. For simplicity sake. I take coefficients to squares thus we are given the metric

$$ds^2 = E^2du^2 + G^2dv$$

Where $E(u,v), G(u,v) > 0$ are smooth functions in U. It is natural to guess that

$$w^1 = Edu, \quad w^2 = Gdv$$

Then, differentiating

$$dw^1 = E_v dv \wedge du = w^2 \wedge w^1_2 = Gdv \wedge \frac{E_v}{G} du$$

$$dw^2 = G_v du \wedge dv = w^1 \wedge w^2_1 = Edu \wedge \frac{G_u}{E} dv$$

Thus we may take

$$w^2_1 = -\, w^1_2 = \frac{G_u}{E} dv - \frac{E_v}{G} du$$

Hence by differentiating again

$$-kw^1 \wedge w^2 = -kEGdu \wedge dv = dw^2_1 = (\frac{\partial}{\partial u}\left(\frac{G_u}{E}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{G}\right))du \wedge dv$$

Form which it follows that

Volume : 6 | Issue : 5 | May 2016 | ISSN - 2249-555X | IF : 3.919 | IC Value : 74.50

$$k = \frac{1}{EG}\left(\frac{\partial}{\partial u}\left(\frac{G_u}{E}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{G}\right)\right)$$

3. Orthogonal separation of variables

For simplicity we consider the geodesic Hamiltonian defined by

$$H_g = \frac{1}{2} g^{ij} p_i p_j \tag{10}$$

The $n-1$ first integrals $F_1, \dots, F_{n-1}$ are given by

$$F_r = \frac{1}{2} K_r^{ij} p_i p_j \qquad r = 1, \dots, n-1$$

that satisfy

$$\{H_g,\ F_r\} = 0, \qquad r = 1, \dots, n-1$$

This implies that

$$[g, k\,] = 0$$

Which is equivalent to

$$K_{r(ab,c)} = 0 \qquad r = 1, \dots, n-1$$

The characterization of orthogonal separability interms of the single killing tensor was obtained by Benenti [14] via the following

3.1 Theorem (Benenti)

A Hamiltonian system defined by (10) is orthogonally separable if and only if there exists a valence two killing tensor K with pointwise simple and real eigenvalues, orthogonally integrable eigenvectors and such thatd $(\tilde{K} dV) = 0$, where the linear operator $\tilde{K}$ is given by $\tilde{K} := Kg$ (or in the index for$\tilde{K}^i_j := K^{il} g_{lj}$) .

We also have the following criterion for orthogonal separation with respect to Cartesian coordinates.

Volume : 6 | Issue : 5 | May 2016 | ISSN - 2249-555X | IF : 3.919 | IC Value : 74.50

2.2 Theorem

The Hamiltonian system (10) is orthogonally separable with respect to cartesian coordinates if the associated pseudo-Riemannian manifold $(\widetilde{\mathrm{M}}, \mathrm{g})$ admits a valence two covariant killing tensor K with pointwise simple eigenvalues and vanishing Nijenhuis tensor $\mathrm{N}_{\widetilde{\mathrm{K}}}$.

We now illustrate the above theorem in two –dimensional Riemannian manifold.so theorem (10) may be treated as follows:

$$\text{Let } g_{ab} = \delta_{ab} E^a \otimes E^b \tag{11}$$

$$\text{And } k_{ab} = \lambda_a \delta_{ab} E^a \otimes E^b \tag{12}$$

Where $\otimes$ is the symmetric tensor product and $a, b = 1, 2$ and λ_1, λ_2 along with the dual vectors , E_1, E_2 are the eigenvalues and eigenvectors of k, then we have two independent connection coefficients Γ_{112}, Γ_{212} and one component of the Riemannian curvature tensor R_{1212} for convenience we have $\alpha \coloneqq \Gamma_{112}$, $\beta \coloneqq \Gamma_{212}$ then we have

$$[\mathrm{E}_1, \mathrm{E}_2] = \alpha \mathrm{E}_1 - \beta \mathrm{E}_2 \tag{13}$$

$$\mathrm{dE}^1 = \alpha \mathrm{E}^1 \wedge \mathrm{E}^2, \quad \mathrm{dE}^2 = \beta \mathrm{E}^1 \wedge \mathrm{E}^2, \tag{14}$$

$$\mathrm{R}_{1212} = -\mathrm{E}_1 \beta + \mathrm{E}_2 \alpha - \alpha^2 - \beta^2, \tag{15}$$

$$\mathrm{E}_1 \lambda_1 = 0, \ \ \mathrm{E}_2 \lambda_1 = 2\alpha(\lambda_2 - \lambda_1), \ \ \mathrm{E}_1 \lambda_2 = 2\beta(\lambda_2 - \lambda_1), \mathrm{E}_2 \lambda_2 = 0 \tag{16}$$

where (4.14) has been used. Our next observation is that in a two-dimensional Riemannian manifold the conditions of orthogonal intergrability for E_1 and E_2,

$\mathrm{E}^{\mathrm{a}} \wedge \mathrm{dE}^{\mathrm{a}} = 0$, $\mathrm{a} = 1,2$ are automatically satisfied. Hence, by Frobenius’ theorem, there exist functions f, g, u and v such that

$$\mathrm{E}^1 = f du, \quad \mathrm{E}^2 = \mathrm{g} dv. \tag{17}$$

we choose $(u\,,v)$ as coordinates, while the functions f and g remain to be determined by the condition of problem. Clearly with respect to (u, v) we have $\alpha = \alpha(u\,, v), \beta = \beta(u\,, v)$ and the eigenvectors E_1, E_2 of K are given by

$$E_1 = (f)^{-1}\,\partial_u, \quad E_2 = (g)^{-1}\,\partial_{v.} \tag{18}$$

substituting (2.9) into (2.3), yields

$$\alpha = -(f g)^{-1}\,\partial_u f, \quad \beta = (f g)^{-1}\,\partial_v g \tag{19}$$

The position - mometa in natural coordinates is Hamiltonian function (2.1)

$$H = \tfrac{1}{2} g^{ab} p_a p_b + V \tag{20}$$

where $g^{ab} = g^{ij} h_i^a h_j^b$ and $p_a = h_a^k p_{k,}$ where h_a^i is defined in (4.12) and V is a function of u and v. next we apply the vector field $[E_1, E_2]$ to λ_1 and λ_2 to obtain the following integrability conditions :

$$E_1\alpha = -3\alpha\beta, \tag{21}$$

$$E_2\beta = 3\alpha\beta, \tag{22}$$

Now it is natural to analyze the following three cases defined with respect to α and β.

(1) $\alpha = \beta = 0 \Leftrightarrow \lambda_1$ and λ_2 constant,

(2) $\alpha = 0, \beta \neq 0 (\alpha \neq 0, \beta = 0) \Leftrightarrow \lambda_1$ constant (λ_2 constant),

(3) $\alpha\beta \neq 0 \Leftrightarrow \lambda_1$ and λ_2 both non $-$ constant.

This classification is intrinsic since the rigid moving frame we are using is defined up to a sign. The general forms of the separable metric

$$ds^2 = (E^1)^2 + (E^2)^2, \tag{23}$$

and the corresponding killing tensor K (2.3) will be derived in each case. Having found the killing tensor, we shall derive the form of the most general separable potential $V(u,v)$ admitted by original Hamiltonian (2.1). To accomplish this, we take into consideration the condition $d(BdV) = 0$ of theorem (2.1) which may be written in terms of the moving frame as

$$E_1E_2V + 3\beta E_2V - 2\alpha E_1V = 0 \tag{24}$$

Once the potential V is found, we derive the second first integral of the Hamiltonian system defined by (10) given by $F = K^{ab}p_ap_b + U$ or

$$F(u,v,p_1,p_2) = \lambda_1 p_1^2 + \lambda_2 p_2^2 + U(u,v) \tag{25}$$

In the moving frame, by solving the equation $dU = 2BdV$. writing this condition in the moving frame, we immediately obtain the following system

$$E_1U = 2\lambda_1 E_1V, \tag{26}$$

$$E_2U = 2\lambda_2 E_2V, \tag{27}$$

We shall consider the following characterization

(1) $\alpha = \beta = 0$

In this case we get from (2.10) that $f = f(u)$and g =g(v) therefore, $E^1 = f(u)du$, $E^2 = g(v)dv$, and the metric takes the form

$$ds^2 = f^2(u)du^2 + g^2(v)dv^2$$

We observe that there exist coordinate transformations $(u,v) \to (\tilde{u},\tilde{v})$,such that

$$E^1 = f(u)du = d\tilde{u}, \quad E^2 = g(v)dv = d\tilde{v} \tag{28}$$

Where

$$\tilde{u} = \int \mathrm{f}(u)\mathrm{d}u, \quad \tilde{v} = \int \mathrm{g}(v)\mathrm{d}v.$$

The remaining coordinate freedom is

$$\tilde{u} = \tilde{u} + u_0, \quad \tilde{v} = \tilde{v} + v_{0.}$$

Thus, for case (1) we have

$$\mathrm{E}^1 = \mathrm{d}u, \quad \mathrm{E}^2 = \mathrm{d}v \tag{2.20}$$

. Thus, the metric has the form

$$\mathrm{ds}^2 = \mathrm{d}u^2 + \mathrm{d}v^2 \tag{2.21}$$

We conclude that the separable coordinates in this case are Cartesian. We also observe, by (2.6), that $\mathrm{R}_{1212} = 0$, in case (1). The eigenvalues of K are constant is compatible with only a flat two-dimensional Riemannian space. The killing equations render when $\lambda_1 = \mathrm{c}_1$ and $\lambda_2 = \mathrm{c}_2$

$$\mathrm{K} = \mathrm{diag}(\mathrm{c}_1, \mathrm{c}_2) \tag{2.22}$$

And in view of (2.15), we have

$$\mathrm{V}(\mathrm{u}, v) = \mathrm{V}_1(\mathrm{u}) + \mathrm{V}_2(v) \tag{2.23}$$

Similarly, by making use of (2.17) and (2.18), we find the corresponding U to be

$$\mathrm{U}(\mathrm{u}, v) = 2\mathrm{kV}_1(\mathrm{u}) + 2\mathrm{LV}_2(v) \tag{2.24}$$

We conclude that a second first integral F that is functionally independent of the Hamiltonian H is

$$\mathrm{F}(\mathrm{u}, v, \mathrm{p}_\mathrm{u}, \mathrm{p}_v) = \mathrm{p}_v^2 + 2\mathrm{V}_2(v) \tag{2.25}$$

We note that the class of Hamiltonian systems just described has the properties of being bi-Hamiltonian in the separable coordinates (u, v) with respect to the constant Poisson bi-vectors P_0 and P_1 :

$$P_0 = \partial_u \wedge \partial_{p_u} + \partial_v \wedge \partial_{p_v}, \quad P_1 = \partial_u \wedge \partial_{p_u} - \partial_v \wedge \partial_{p_v}, \tag{2.26}$$

and having a Lax representation defined by matrices L and M of the form

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \qquad M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \tag{2.27}$$

where

$$L_i = \begin{pmatrix} \frac{1}{\sqrt{2}} p_j & 2\omega_j \\ \frac{f_i(\omega_j)}{\omega_j} & -\frac{1}{\sqrt{2}} p_j \end{pmatrix}, \qquad M_i = \frac{1}{2\omega_j} \begin{pmatrix} 0 & 0 \\ \frac{d}{dt}\left(\frac{p_j}{\sqrt{2}}\right) & -2p_j \end{pmatrix}, \tag{2.28}$$

where $i, j = 1,2, i \neq j, \omega_1 = u, \omega_2 = v$ and $f_1, f_2 \in C^1(\mathbb{R})$ are arbitrary functions.

(2) $\alpha = 0, \beta \neq 0$ $(\alpha \neq 0, \beta = 0)$

We have in this case

$$ds^2 = du^2 + g^2(u) dv^2 \tag{2.29}$$

Where g(u) is an arbitrary function. To solve the killing equation and find the corresponding K, we observe that in view of the above $\beta = \partial_u g/g$ .now (2.7) transform into the following system of partial differential equations

$$\partial_u \lambda_1 = \partial_v \lambda_1 = \partial_v \lambda_2 = 0, \quad \partial_u \lambda_2 = \partial_u g g^{-1}(\lambda_2 - \lambda_1), \tag{2.30}$$

Solving for λ_1 and λ_2 we find $\lambda_1 = k$, $\lambda_2 = Lg^2(u) + k$, where L, k are arbitrary constants. Hence, the killing tensor in this case takes the form:

$$K = \mathrm{diag}(k, Lg^2(u) + k) = kg + LK_1, \tag{2.31}$$

where $K_1 = \mathrm{diag}(0, g^2(u))$ and g, K_1, span two- dimensional Abelian Lie alge-bra of killing tensors as in [4].

3. Some applications

We now apply to the Hamiltonian systems defined in particular Riemannian spaces .

3.1 ***Two-dimensional Euclidean space E^2***

In this case $R_{1212} = 0$, which entails

$$E_2\alpha - E_1\beta = \alpha^2 + \beta^2.$$

Consider now the following three separable cases, defined with respect to the functions α and β.

(1) $\alpha = \beta = 0$.

In this case the separable coordinates are obviously Cartesian and $R_{1212} = 0$, is automatically satisfied.

(2) $\alpha = 0,\ \beta \neq 0\ (\alpha \neq 0,\ \beta = 0)$

Solving Eq. (3.36) we obtain that the metric can be written as follows:

$$ds^2 = du^2 + u^2 dv^2, \tag{3.1}$$

3.2 Surfaces of rotation

A surface of rotation is the surface generated by the rotation of a plane curve C around an axis in its plane. If C is parameterized by the equations $\rho = \rho(u)$ and $z = z(u)$, the position vector of the surface of rotation is $\mathbf{r} = \{\rho(u) \cos v, \rho(u) \sin v, z(u)\}$, where u is the parameter of the curve C, ρ is the distance between a point on the

Volume : 6 | Issue : 5 | May 2016 | ISSN - 2249-555X | IF : 3.919 | IC Value : 74.50

surface and the axis z of rotation and v is the angle of rotation, which is the ignorable (cyclic) coordinate. The metric of the surface of rotation is

$$ds^2 = ((\rho')^2 + (z')^2)du^2 + \rho^2 dv^2. \tag{3.2}$$

Clearly, the metric (3.2) can be reduced to the form (3.1) by an appropriate coordinate transformation. Once the curvature $R_{1212}(u)$ is known, the function $g(u)$ and the corresponding metric may be recovered from (3.36) and vice versa. Consider an example. The metric

$$ds^2 = a^2 du^2 + \backslash \ell^2 \left(1 + \frac{a}{\ell}\cos u\right)^2 dv^2 \tag{3.3}$$

defines the surface of a two-dimensional torus T^2, where a and ℓ are the radii of the rotating and axial circles, respectively. We note that in this paper we do not consider global properties of two-dimensional pseudo-Riemannian manifolds, hence here T^2 is not a topological torus. Locally, the metric (2.6) yields one system of separable coordinates with $g(u) = \ell(1 + (a/\ell)\cos(u/a)$, $R_{1212} = \cos(u/a)/(a\ell + a\cos(u/a))$ and the other quantities as in Case (2) of Section 3 corresponding to the given $g(u)$.

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