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Prime Join Matrices and The Reciprocal S-Prime Join Matrices on Posets

KEYWORDS: Lattice, S- prime Join, arithmetical functions and Mobius function

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ABSTRACT *We consider S-prime Join matrices as an abstract generalization of S-prime Least common multiple matrices . We also found determinant and inverse of both S- prime and reciprocal S-prime join matrices are discussed and also discussed some of the most important properties of S-prime Join matrices are presented interms of S- prime Join matrices .*

INTRODUCTION

Let S = $\{x_1, x_2, ..., x_n\}$ be a set of n positive integers with $x_1 < x_2 < ... < x_n$ and let $f : P \to \mathbb{C}$ be a complex valued function on Z+ (i.e., arithmetic function).Let (x_i, x_j) denotes the greatest common divisor (gcd) of x_i and x_j and defines the $n \times n$ matrix $(S)_f$ by $((S)_f)_{ij}$ = $f(x_i, x_j)$ .We refer to $(S)_f$ as the GCD Matrix on S with respect to f .The Set S is said to be gcd-closed if $(x_i, x_j) \in S$ whenever $x_i, x_j \in S$ .The set S is said to be factor-closed if it contains every positive divisor of each $x_i \in S$ .Clearly, a factor-closed set is always gcd– closed but the converse does not hold.Let $[x_i, x_j]$ denotes the least common multiple (lcm) of x_i and x_j and defines the $n \times n$ matrix $[S]_f$ by $([S]_f)_{ij}$ = $f[x_i, x_j]$ .We refer to $[S]_f$ as the LCM Matrix on S with respect to f .The set S is said to be lcm-closed if $[x_i, x_j] \in S$ whenever $x_i, x_j \in S$ .The set S is said to be multiple-closed if it is lcm-closed and $x_i \mid d \mid x_n \Rightarrow d \in S$. Here | stands for the usual divisibility relation of integers.

In 1876, the concept of Classical Smith determinant with entries on Z+ was introduced by H.J.S. Smith [12] is

$$\det[(x_i, x_j)]_{n \times n} = \phi(x_1).\phi(x_2).\phi(x_3)..\phi(x_n).$$

H.J.S.Smith also calculated the determinant of the LCM Matrix on a factor – closed set.

In 1876, H.J.S. Smith results extended L.E.Dickson proved that if α_{ij} = $(i, j);\ i, j = 1,2,3,...r$ then $\det((\alpha_{ij}) = \varphi(1)\varphi(2)...\varphi(r)$ where φ is the Euler φ function.

In 1991, S.Beslin [3,4,5] defined the LCM Matrix, which is an $n \times n$ matrix whose i,j – entry is the least common multiple of x_i, x_j. In 1992, K.Bourque and S.Ligh [6,7,8] proved that the LCM Matrix [S] is nonsingular if S is Factor Closed set. They also conjectured that the LCM Matrix [S] is non singular if S is GCD closed.

In 2003, A.A. Oval established various results concerning GCD Matrices and Least Common Multiple (LCM) Matrices. In 1968, Wilf has proved, let $f : P \to R$ where P a Meet semi –lattice and let M be the matrix with $M_{X,Y} = f(X \wedge Y)$ then $\det(M) = \prod_{Z \in P} g(Z)$ where $g(Z) = \sum_{W \le Z} \mu(W, Z) f(W)$.

In 1960, L.Carlitz [9], gave a new form of gcd-matrices and determinant value, $[f(i,j)]_n$ = C (diag(g(1),…,g(n))) C^T where $C = (C_{ij})_{nxn}$;

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$C_{ij} = \begin{cases} 1 & \text{if } j \nmid i \\ 0 & \text{if } j \mid i \end{cases}$ and

$D = (d_{ij})$ diagonal matrix

$d_{ij} = \begin{cases} g(i) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$\therefore \det [f(i,j)]_{nxn} = g(1).g(2)\dots g(n)$

2.The origin of the Join Matrices on Posets

In this section, we define preliminary concepts that are needed to understand the summaries of the articles in this section.

Definition :2.1

Let $(P, \preceq) \subseteq (Z^+, |)$ be a partially ordered set. We call P a Join-semi lattice if for any $x,y \in P$ there exists a unique $z \in P$ such that

(i) $x \leq z$ and $y \leq z$ and
(ii) If $x \leq w$ and $y \leq w$ for some $w \in P$ then $z \leq w$

In such a case z is called the Join of x and y and it is denoted by $x \vee y$.
A Join semi-lattice, which is also a Meet-Semi lattice, is called a lattice.

Definition :2.2

Let $(P, \preceq, \wedge)$ be a Meet –Semi lattice and defined the partial order $\preceq$ on P by $x \preceq y \Leftrightarrow y \preceq x$. Then for any $x,y \in P$ there exists a unique $z = x \vee y = x \wedge y$ that satisfies (i) and (ii) above for $\preceq$. Thus $(P, \preceq, \vee)$ is a Join-semi lattice and it is said to be the dual of $(P, \preceq, \wedge)$.

Definition: 2.3

Let $(P, \preceq, \wedge, \vee)$ be a lattice in which every principal order ideal is finite. Let $S = \{x_1, x_2, ..., x_n\}$ be a subset of P such that $x_i \preceq x_j \Rightarrow i \leq j$ and let $f : P \to \mathbb{C}$ be a function. Then the n x n matrices $[S]_f$ are defined by $[S]_f = \lfloor f(x_i \vee x_j) \rfloor$ is called the Join Matrix on S associated with f.

Definition: 2.4

In this lattices S is a finite set of positive integers, $f : P \to C$ is an arithmetical function and the S – prime Join Matrix is called LCM Matrices, which is defined as $([S]_f)_{ij} = f(lcm(x_i, x_j))$.

Definition: 2.5

If $(P, \preceq) \subseteq (Z^+, |)$ then the Join Matrices respectively LCM Matrices on S.The set S is said to be upper-closed if for every $x,y \in P$ with $x \in S$ and $x \preceq y$ we have $y \in S$.The set S is said to be Join – closed if for every $x, y \in S$.We have $x \vee y \in S$.

Definition:2.6

We say that f is a semi-multiplicative function on P, if $f(x \wedge y) f(x \vee y) = f(x) f(y)$ for all $x, y \in P$.

Definition :2.7

Let x and y be the two elements of the poset P and μ is the mobius function of the poset $(S, \prec)$ then

$$\mu(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \\ -\sum_{z: z \leq y} \mu(x,z) & \text{otherwise} \end{cases}$$

Lemma:2.8

Let g be an incidence function of P.Then $g(x,y) = \sum_{x \leq z \leq y} (\mu * g)(z,y)$ for all $x,y \in P$.

Lemma:2.9

Let $\uparrow S = \{w_1, w_2, w_3, ..., w_r\}$ with $w_i < w_j \Rightarrow i < j$ and let A denote the $n \times n$ matrix defined by

$$a_{ij} = \begin{cases} \sqrt{(\mu * f_u)(w_j, 1)} & if \quad x_i \le w_j \\ 0 & otherwise \end{cases}$$

Then $[S]_f = AA^T$.

Proof:

For $1 \le i \le n$, $1 \le j \le r$ we have

$$(AA^T)_{ij} = \sum_{k=1}^{r} a_{ik} a_{jk} =$$

$$\sum_{\substack{x_i \le w_k \le 1 \\ x_j \le w_k \le 1}} (\mu * f_u)(w_k, 1)$$

$$= \sum_{x_i \vee x_j \le w_k \le 1} (\mu * f_u)(w_k, 1)$$

By lemma (2.8), $(AA^T)_{ij} = f_u(x_i \vee x_j, 1)$

$$= f(x_i \vee x_j)$$

This completes the proof.

Lemma:2.10(Join Matrix in terms of a certain Meet Matrix)

Let $D = \text{diag}(f(x_1), \ldots, f(x_n))$ then $[S]_f = D(S)_{\frac{1}{f}} D$

Proof:

$$Since \left(D(S)_{\frac{1}{f}} D\right)_{ij} = f(x_i)\left((S)_{\frac{1}{f}}\right)_{ij} f(x_j)$$

$$= \frac{f(x_i) f(x_j)}{f(x_i \wedge x_j)} = f(x_i \vee x_j)$$

we have $[S]_f = D(S)_{\frac{1}{f}} D$

3.The Structure of S – Prime Join Matrices on Posets

Definition :3.1

Let $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two subsets of P and the nxn S-Prime Join Matrix on X and Y with respect to f is defined as $M = [X,Y]_f = [f_{ij}]$ where $f_{ij} = \frac{(4x_i + 1)(4y_j + 1)}{4(x_i \wedge y_j) + 1}$.

Definition :3.2

Let $(P, \preceq)$ be a lattice in which every principal order is finite and let f be a complex valued function on P.Let $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two subsets of P.Let the elements of X and Y be arranged so that $x_1 < x_2 < ... < x_n$ and $y_1 < y_2 < ... < y_n$.

Let $D = \{d_1, d_2, ..., d_m\}$ be any subsets of P containing the elements $x_i \vee y_j$; i,j = 1,2,….,n.Let the elements of D be arranged so that $d_1 < d_2 < ... < d_m$.

Definition:3.3

We define $g_{D,\frac{1}{f}}$ on D inductively as

$$g_{D,\frac{1}{f}}(d_k) = f(d_k) - \sum_{d_v < d_k} g_{D,\frac{1}{f}}(d_v)$$

or $f(d_k) = \sum_{d_v \le d_k} g_{D,\frac{1}{f}}(d_v)$ *then*

$$g_{D,\frac{1}{f}}(d_k) = \sum_{d_v \le d_k} \frac{\mu_D(d_v, d_k)}{f(d_v)}$$ where

μ_D is the mobius function on the poset $(D, \preceq)$.

Definition:3.4

Let $E(X) = (e_{ij}(X))$ and $E(Y) = (e_{ij}(Y))$ denotes the $n \times m$ matrices defined by

$$(e_{ij}(X)) = \begin{cases} 1 & if \;\; d_j \le x_i \\ 0 & otherwise \end{cases}$$ and

$$(e_{ij}(Y)) = \begin{cases} 1 & if \;\; d_j \le y_i \\ 0 & otherwise \end{cases}$$

We also denote

$$\Lambda_{D,\frac{1}{f}} = \text{diag}\left(g_{D,\frac{1}{f}}(d_1), ..., g_{D,\frac{1}{f}}(d_m)\right).$$

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Theorem:3.5

If D, E(X), E(Y) and $\Lambda_{D,\frac{1}{f}}$ as defined above then $[X,Y]_f = E(X)\,\Lambda_{D,\frac{1}{f}}\,E(Y)^T$.

Proof :

The i,jth entry in

$$\left(E(X)\Lambda_{D,\frac{1}{f}}E(Y)^T\right)_{ij} = \sum_{k=1}^{n} e_{ik}\Lambda_k e_{kj}$$

$$= \sum_{\substack{x_i \le d_k \\ x_j \le d_k}} g(d_k) = \sum_{[x_i \vee x_j] \le d_k} g(d_k) = [X,Y]_f$$

Hence $[X,Y]_f = E(X)\,\Lambda_{D,\frac{1}{f}}\,E(Y)^T$.

Theorem:3.6

(i) If n>m, then $\det[X,Y]_f = \det[M] = 0$

and (ii) If $n \le m$, then

$$\det(M) = \prod_{i=1}^{n} f(x_i) f(y_i) \times \sum_{1 \le k_1 < k_2 < \ldots < k_n \le m} \det E(X)_{(k1,k2,\ldots kn)} \det E(Y)_{(k1,k2,\ldots kn)}$$

$$\times\, g_{D,\frac{1}{f}}(d_{k_1})\, g_{D,\frac{1}{f}}(d_{k_2}) \ldots g_{D,\frac{1}{f}}(d_{k_n}).$$

Proof:

$[X,Y]_f = D_{x,f}\,(X,Y)_{1/f}\,D_{y,f} = D_{x,f}E(X) \wedge_{D,f} E(Y)^T D_{y,f}$

Also $M = D_{x,f}\tilde{M}\,D_{y,f} \Rightarrow$

$\det M = \det(D_{x,f})\det(\tilde{M})\det(D_{y,f})$

$= \det(\tilde{M})$

$$\prod_{i=1}^{n}[4(x_i+1)]\,[4(y_j+1)]$$

Theorem:3.7

In the case of semi-multiplicative function $M = [X,Y]_f = D_X \tilde{M} D_Y$, where $D_X = \operatorname{diag}(4x_1+1, 4x_2+1, \ldots, 4x_n+1)$, $D_Y = \operatorname{diag}(4y_1+1, 4y_2+1, \ldots, 4y_n+1)$ and $\tilde{M}$ has the entry $\tilde{f}_{ij} = \dfrac{1}{4(x_i \wedge y_j)+1}$.

Proof:

Since $[X,Y]_f = M = [f[x_i \vee y_j]]$

$$= \left[f(x_i)\frac{1}{f(x_i \wedge y_j)}f(y_j)\right] = D_X\tilde{M}D_Y$$

Theorem:3.8

If D is Meet closed and f and g are arithmetical functions then

(i) $f(d_k) = \sum_{z \le d_k} g_{D,f}(d_k)$

implies and implied by

(ii) $g_{D,f}(d_k) = \sum_{\substack{z \le d_k \\ z \le d_t \\ t<k}} \sum_{w \le z} f(w)\mu_P(w,z)$

Proof:

Assume (i) and prove (ii)

Consider $\sum_{z \le d_k} \mu(z) = \sum_{z'z=d_k} f(z')\mu(z)$

$$= \sum_{z'z=d_k} \mu(z)\sum_{e \le z'} g(e) = \sum_{eh'z=d_k} \mu(z)g(e)$$

$$= \sum_{eh'z=d_k} g(e)\sum_{z \le h'} \mu(z)$$

Since the sum $\sum_{z \le h'} \mu(z)$ has the value 0 if h'>1 and the value 1 if h' = 1.

Hence $\sum \mu(z) f\left(\dfrac{d_k}{z}\right) = g(d_k)$

To prove the converse: we consider

$$\sum_{z \le d_k} g(z) = \sum_{z \le d_k}\sum_{z' \le d_k} \mu(z') f\left(\frac{z}{z'}\right)$$

$$= \sum_{eh'=d_k} \mu(z') f(e) = \sum_{eh'=d_k} f(e) \sum_{z' \le h'} \mu(z')$$

As before, the sum of $\sum_{z' \le h'} \mu(z')$ has the value 0 if h'>1 and the value 1 if h'=1.

Hence $\sum_{z \le d_k} g(d_k) = f(d_k)$.

Theorem:3.9

If D is Join-closed set then

$$g_{D,\frac{1}{f}}(d_k) = \sum_{z \le d_k} \sum \frac{\mu(w,z)}{f(w)} \text{ where } \mu \text{ is}$$

the mobius function of P.

Proof:

It is similar to the proof of the theorem(3.8).

Theorem:3.10$(X,Y)_f = D_X\ [X,Y]_{1/f}\ D_Y$.

Proof:

Now we consider the example

S = {1,2} and T ={2,3}

D_X =diag(5,9) and D_Y = diag(9,13)

$(S)_f = (X,Y)_f =$

$$\begin{pmatrix} 4(1\wedge 2)+1 & 4(1\wedge 3)+1 \\ 4(2\wedge 2)+1 & 4(2\wedge 3)+1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 5 \\ 9 & 5 \end{pmatrix}$$

$[X,Y]_{1/f}$ =

$$\begin{pmatrix} \frac{4(1\wedge 2)+1}{(4(1)+1)(4(2)+1)} & \frac{4(1\wedge 3)+1}{(4(1)+1)(4(3)+1)} \\ \frac{4(2\wedge 2)+1}{(4(2)+1)(4(2)+1)} & \frac{4(2\wedge 3)+1}{(4(2)+1)(4(3)+1)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{9} & \frac{1}{13} \\ \frac{1}{9} & \frac{5}{117} \end{pmatrix}$$

$D_X\ [X,Y]_{1/f}\ D_Y$ =

$$\begin{pmatrix} 5 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} \frac{1}{9} & \frac{1}{13} \\ \frac{1}{9} & \frac{5}{117} \end{pmatrix} \begin{pmatrix} 9 & 0 \\ 0 & 13 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 5 \\ 9 & 5 \end{pmatrix}$$

Hence Proved.

Theorem:3.11

Let S={$x_1, x_2, x_3, \ldots, x_n$ } be S-prime Join -closed . Without loss of generality we may assume that i<j whenever $x_i < x_j$, then

$$g_{s,f}(x_j) = \sum_{\substack{z \le x_j}} \sum_{\substack{w \le z \\ z \le x_t \\ t<j}} f(w)\mu(w,z) \text{ where } \mu \text{ is}$$

the mobius function of P.

Proof:

By using the definition (3.3)

$$f(x_j) = \sum_{x_i \le x_j} g_{s,f}(x_i) = \sum_{x_i \le x_j} \sum_{\substack{z \le x_j \\ z \not\le x_t \\ t<j}} \sum_{w \le z} f(w)\mu(w,z)$$

We write,

$$f(x) = \sum_{z \le x} g(z) \ or\ g(x) = \sum_{z \le x} f(z)\mu(z,x)$$

for all $x \in P$

It has to be prove that,

$$\sum_{z \le x_j} g\ (z) = \sum_{x_i \le x_j} \sum_{\substack{z \le x_i \\ z \not\le x_t \\ t<i}} g(z)$$

Now consider the sum of R.H.S of equation (1)

Let $x_i \le x_j$ and $z \le x_i \Rightarrow z \le x_j$.

Thus every z occurring on the right side of equation (1) ccurs on the left side of equation (1).

Conversely, Consider the sum on the left side

of equation (1).

Suppose that $z \le x_j$ we have $z \le x_i$ by minimality

of i, we have r = i or $x_r = x_i$,therefore$x_r \le x_j$ means $x_r \le x_j$ thus every z occurring on the side

of equation (1).

This completes the proof .

Theorem:3.12

If S is lower closed subset of P

then $g_{s,f}(x_j) = \sum_{x_i \le x_j} f(x_i)\mu(x_i,x_j)$

Proof:

Already we know that the result,

$$g_{s,f}(x_j) = \sum_{\substack{z \le x_j \\ z \not\le x_t \\ t<j}} \sum_{w \le z} f(w)\mu(w,z)$$

It reduces we get the proof of theorem.

Then S is lower closed.

Example :3.13

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Let $S=\{x_1,x_2,\ldots\ldots x_n\}$ be a chain with $x_1<x_2<\ldots\ldots<x_n$. Then $g_{s,f}(x_1)=f(x_1)$, $g_{s,f}(x_2)=f(x_2)-f(x_1)$

In general

$g_{s,f}(x_j)=f(x_j)-f(x_{j-1})$ where,

j=2, 3, 4,…, n.

Example:3.14

Let $S=\{x_1,x_2,\ldots..x_n\}$ be an incomparable set and let $S=\{x_0, x_1, x_2, \ldots..x_n\}$. Then,

$$g_{s,f}(x_0)=f(x_0),$$

$$g_{s,f}(x_1)=f(x_1)-f(x_0)$$

and $g_{s,f}(x_2)=f(x_2)-f(x_0)$.

In general $g_{s,f}(x_j)=f(x_j)-f(x_0)$ for j=1, 2, 3, …,n

Theorem :3.15

Let $S=\{x_1, x_2\ldots x_n\}$ and $T=\{y_1, y_2\ldots y_m\}$ be any two subsets of P. Define the incidence matrix whose i, j-entry is 1 if $y_j \le x_i$ and zero otherwise namely that is, E(S, T) $=(e_{ij})_{n\times m}$ where

$$(e_{ij})=\begin{cases}1 & ,if \quad y_j\le x_i\\ 0 & ,if \quad otherwise\end{cases}$$

Theorem: 3.16

If S is a S-Prime join-closed.Then $\det[S]_f = \prod_{i=1}^{n} g_{s,f}(x_i)$.

Proof:

The theorem is proved and verified with a suitable example.

Consider the set S = {1,2,3}

$$Then \quad [S]_f=\begin{bmatrix} f(5) & f(9) & f(13)\\ f(9) & f(9) & f25)\\ f(13) & f(25) & f(13)\end{bmatrix}$$

$Let \ \det[S]_f =$

$$f(5)[f(9)f(13)-f(25)^2]-f(9)[f(9)f(13)-f(13)f(25)]+f(13)[f(9)f(25)-f(13)f(9)]$$

$$=f(5)f(9)f(13)-f(5)f(25)^2-[f(9)^2f(13)]+[f(9)f(13)f(25)]+[f(9)f(13)f(25)-f(13)^2f(9)]\ldots(1)$$

By using example,

$$g(9) = f(9)-f(5);$$
$$g(13) = f(13)-f(9);$$
$$g(25) = f(25)-f(13);$$

$$\prod_{i=5,9,13}^{n}(g(x_i)) = g(x_1)g(x_2)g(x_3)$$

$$=[f(9)-f(5)][f(13)-f(9)][f(25)-f(13)]$$

$$=f(5)f(9)f(13)-f(5)f(25)^2-[f(9)^2f(13)]+[f(9)f(13)f(25)]+[f(9)f(13)f(25)-f(13)^2f(9)]\ldots(2)$$

From equation (1) and (2), we get;

$$\det[S]_f = \prod_{i=1}^{n} g_{s,f}(x_i).$$

Hence the theorem is proved.

Corollary : 3.17

If $S=\{x_1, x_2, x_3,\ldots,x_n\}$ is a chain with $x_1<x_2<x_3\ldots<x_n$. Then

$$\det[S]_f= f(x_1)\prod_{i=1}^{n}[f(x_i)-f(x_{i-1})]$$

Proof:

By using theorem,

If S is a S-prime Join –closed then $\det[S]_f=\prod_{i=1}^{n}(g_{s,f}(x_i))$ and the result ,

= f (5) f (9)f (13)+f $(5)^{3--}$ f$(5)^2$f(13)- f$(5)^2$f(9)

We have ,

det $[S]_f$= f (5) [f (9)-f (5)][f(13)-f(5)]

det $[S]_f$ = g(1) g(2) g(3)

Then, det $[S]_f= f(x_1)\prod_{i=1}^{n}[f(x_i)-f(x_{i-1})]$

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Hence Proved.

Theorem:3.18

Let T ={ $y_1, y_2, y_3, \ldots, y_m$ } be a S-prime Join–closed subset of P containing S={ $x_1, x_2, x_3, \ldots, x_n$ }. Then,

det $[S]_f$ =

$$\sum_{1\le k_1\le..\le k_n\le m} \det[E(k_1,k_2,\ldots,k_n)^2 g_{T,f}(y)_{k1}, g_{T,f}(y)_{k2},\ldots g_{T,f}(y)_{kn}]$$

Where, E=E(S,T)

Proof:

$[S]_f = E \Lambda E^T$, also det(E) = det(E^T), by using known theorem.

Now we consider the example ,

S = {2,3} and T ={1,2,3} .Then,

$$[S]_f = [f(4(x_i \vee x_j)+1)] = \begin{bmatrix} f(4(2\vee 2)+1) & f(4(2\vee 3)+1) \\ f(4(3\vee 2)+1) & f(4(3\vee 3)+1) \end{bmatrix}$$

$$[S]_f = \begin{bmatrix} f(9) & f(25) \\ f(25) & f(13) \end{bmatrix}$$

The incident matrix of S&T is,

$$E = E(S,T) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$E \Lambda E^T$=

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} f(5) & 0 & 0 \\ 0 & f(9)-f(5) & 0 \\ 0 & 0 & f(13)-f(5) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} f(9) & f(5) \\ f(5) & f(13) \end{bmatrix}$$

$\therefore [S]_f = E \wedge E^T$

Also, det (E)

$$\Rightarrow E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 0$$

$$\det(E^T) \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = 0$$

det (E) = det (E^T)

det $[S]_f$ =

$$\sum_{1\le k_1\le..\le k_n\le m} \det[E(k_1,k_2,\ldots,k_n)^2 g_{T,f}(y)_{k1}, g_{T,f}(y)_{k2},\ldots g_{T,f}(y)_{kn}]$$

Hence proved.

Theorem:3.19

If X and Y are Meet Closed and Lower Closed sets then

$$\det(M) = \prod_{i=1}^{n} f(x_i) f(y_i) g(d_i)$$

where $g(d_i) = \sum_{d_j \le d_i} \frac{\mu(d_j, d_i)}{f(d_j)}$.

Proof:

To get the proof, by using the theorem (3.6) and (3.7).

Theorem:3.20

Let $X_i = X\backslash\{x_i\}$ and $Y_i = Y\backslash\{y_i\}$ for i = 1,2,3,…,n.If M is invertible then the inverse of M is the $n \times n$ matrix B = (b_{ij}) where $b_{ij} = \frac{\alpha_{ji}}{\det(M)}$,

where α_{ji} is the co-factor of ji-entry of M.

Proof:

It is a general method used to prove.

Theorem:3.21

Let $X_i = X\backslash\{x_i\}$ and $Y_i = Y\backslash\{y_i\}$ for i = 1,2,3,…,n.If M is invertible then the inverse of M is the $n \times n$ matrix B = (b_{ij}), where

$$b_{ij} = \frac{(-1)^{i+j}}{f(x_j) f(y_i) \det(M)} \prod_{v=1}^{n} f(x_v) f(y_v)$$

$$\times \sum_{1\le k_1<k_2<\ldots<k_n\le m} \det E(X_j)_{(k1,k2,\ldots,kn-1)} \det E(Y_i)_{(k1,k2,\ldots,kn-1)}$$

$$\times\, g_{D,\frac{1}{f}}(d_{k_1})\, g_{D,\frac{1}{f}}(d_{k_2})\ldots g_{D,\frac{1}{f}}(d_{k_{n-1}})$$

)

Proof:

Since $b_{ij} = \dfrac{\alpha_{ji}}{\det(M)}$, where α_{ji} is the co-factor of ji-entry of M.

It is easy to see that

$\alpha_{ji} = (-1)^{i+j} \det[X_j, Y_i]_f$.

By theorem (3.6) we see that

$\det[X_j, Y_i]_f =$

$$\sum_{1\le k_1<k_2<\ldots<k_{n-1}\le m} \det E(X_j)_{(k1,k2,\ldots,kn-1)} \det E(Y_i)_{(k1,k2,\ldots,kn-1)}$$

$$\times\, g_{D,\frac{1}{f}}(d_{k_1})\, g_{D,\frac{1}{f}}(d_{k_2})\ldots g_{D,\frac{1}{f}}(d_{k_{n-1}})$$

combining the above equations we obtain the theorem.

Example:

Construct the 2 x 2 S-Prime Join Matrix on the LCM closed sets X= {1,2} and Y ={2,5}.Then by using the definition (2.7),

M =[f_{ij}] where $f_{ij} = \dfrac{(4x_i+1)(4y_j+1)}{4(x_i \wedge y_j)+1}$

By using the definition of f and $\mu(x,y)$ we obtain; $f_{11} = 9$, $f_{12} = 21$, $f_{21} = 9$, $f_{22} = 189/5$

$$\therefore M = \begin{pmatrix} 9 & 21 \\ 9 & \frac{189}{5} \end{pmatrix}$$

Since det(M) = $\prod_{i=1}^{n} f(x_i) f(y_i) g(d_i)$ where $g(d_i) = \sum_{d_j \le d_i} \dfrac{\mu(d_j, d_i)}{f(d_j)}$

here D ={1,2} each $d_i \in x_i \wedge y_j$ for i,j = 1,2.

$f(x_1) = f(1) = 5$, $f(x_2) = f(2) = 9$,
$f(y_1) = f(2) = 9$, $f(y_2) = f(5) = 21$

$g(d_1) = g(1) = \sum_{d_j\le 1} \dfrac{\mu(d_j,1)}{f(d_j)} = \dfrac{\mu(1,1)}{f(1)} = \dfrac{1}{5}$

$g(d_2) = g(2) = \sum_{d_j\le 2} \dfrac{\mu(d_j,2)}{f(d_j)} = \dfrac{\mu(1,2)}{f(1)} + \dfrac{\mu(2,2)}{f(2)} = \dfrac{-4}{45}$

Thus $\det(M) =$ f(1)f(2)f(2)f(5)g(1)g(2) $= -\dfrac{756}{5}$

Find M^{-1}, by using the theorem, B = (b_{ij}) where $b_{ij} = \dfrac{(-1)^{i+j}}{\det(M)} M_{ji}$

$b_{11} = \dfrac{(-1)^{1+1}}{\det(M)} M_{11} = -\dfrac{1}{4}$

$b_{12} = \dfrac{(-1)^{1+2}}{\det(M)} M_{21} = \dfrac{5}{36}$

similarly, $b_{21} = 5/84$, $b_{22} = -5/84$

$$M^{-1} = \begin{pmatrix} \frac{-1}{4} & \frac{5}{36} \\ \frac{5}{84} & \frac{-5}{84} \end{pmatrix}$$

4.Inverse of the Reciprocal S-Prime Join Matrices

Definition :4.1

Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two subsets of P and the nxn Reciprocal S - Prime Join Matrix on X and Y with respect to R is defined as R = $[X,Y]_r = [r_{ij}]$ where $r_{ij} = \dfrac{4(x_i \wedge y_j)+1}{(4x_i+1)(4y_j+1)}$.

Definition:4.2

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Let X and Y be two Lower Closed subsets of P then

$$\det(R) = \prod_{i=1}^{n} r(x_i) r(y_i) h(d_i)$$

$$where \quad h(d_i) = \sum_{d_j \leq d_i} \frac{\mu(d_i, d_j)}{r(d_j)}.$$

Definition:4.3

If R is invertible then the inverse of R is nXn matrix H =(h_{ij}) where $h_{ij} = \frac{r_{ji}}{\det(R)}$ and r_{ji} is the cofactor of the ji – entry of R.Therefore, $h_{ij} = \frac{(-1)^{i+j}}{\det(R)} r_{ji}$

Example:

Construct the 2X2 reciprocal S-Prime join matrix on the LCM closed sets X={1,2} and Y = {2,5}.

By using the definition of (2.8),

$$r_{ij} = \frac{4(x_i \wedge y_j)+1}{(4x_i+1)(4y_j+1)}$$

r_{11} =1/9 , r_{12} = 1/21 , r_{21} = 1/9 , r_{22} = 5/189

$$\therefore R = \begin{pmatrix} \frac{1}{9} & \frac{1}{21} \\ \frac{1}{9} & \frac{5}{189} \end{pmatrix}$$

Find det(R) , by using the definition (4.2),

$$\det(R) = \prod_{i=1}^{n} r(x_i) r(y_i) h(d_i)$$

$$where \quad h(d_i) = \sum_{d_j \leq d_i} \frac{\mu(d_i, d_j)}{r(d_j)}$$

and D = {1,2}

r(x_1) = r(1) = 1/5, r(x_2) = r(2) = 1/9,

r(y_1) = r(2) = 1/9, r(y_2) = r(5) = 1/21.

$$h(1) = \sum_{d_j \leq 1} \frac{\mu(d_j,1)}{r(d_j)} = \frac{\mu(1,1)}{r(1)} = 5$$

$$h(2) = \sum_{d_j \leq 2} \frac{\mu(d_j,2)}{r(d_j)} = \frac{\mu(1,2)}{r(1)} + \frac{\mu(2,2)}{r(2)} = -5+9 = 4$$

$$\therefore \det(R) = \frac{1}{5}.\frac{1}{9}.\frac{1}{9}.\frac{1}{21}.5.4 = \frac{4}{1701}$$

Find R^{-1} by using the definition(4.3), we get;

$$h_{11} = \frac{(-1)^{1+1}}{\det(R)} r_{11} = \frac{5/189}{4/9.9.21} = \frac{45}{4}$$

$$h_{12} = \frac{(-1)^{1+2}}{\det(R)} r_{21} = \frac{-81}{4}$$

Similarly, h_{21} = -189/4, h_{22} = 189/4

$$\therefore R^{-1} = \begin{pmatrix} \frac{45}{4} & -\frac{81}{4} \\ -\frac{189}{4} & \frac{189}{4} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 45 & -81 \\ -189 & 189 \end{pmatrix}$$

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