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Some Generalization on Fixed Point Theorems Related to Different Types of Fuzzy Metric Spaces

KEYWORDS: Fuzzy metric spaces, fuzzy 2- metric spaces, fuzzy 3- metric spaces, fuzzy 4- metric spaces, fixed point, Common fixed point.

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ABSTRACT *In the present paper some generalization on fixed point and common fixed point theorems in complete Fuzzy 4-metric spaces are established which are motivated by Gahler [13-15], Sharma, Sharma and Isekey [30], Sharma , S.[31], Shrivastava R., Singhal J.[36].*

1. Introduction:

In 1965, the concept of fuzzy sets was introduced by Zadeh [36]. After that many authors have expansively developed the theory of fuzzy sets and applications. Especially, Deng [8], Erceg [10], Kaleva and Seikhala [23], Karamosil and Michalek [25], have introduced the concept of fuzzy metric spaces in different ways. Recently, many authors [1,6,11,17,20,21,22,27,28,31,32] have also studied the fixed point theory in the fuzzy metric spaces and [2,3,4,5,19,26,33] have studied for fuzzy mappings which opened an avenue for further development of analysis in such spaces and such mappings. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various authors. To use this concept in topology and analysis, many authors have extensively developed the theory of fuzzy sets and applications. With the concept of fuzzy sets, the fuzzy metric space was introduced by I. Kramosil and J. Michalek [8] in 1975. M. Grabiec [5] proved the contraction principle in fuzzy metric spaces in 1988. Moreover, A. George and P. Veeramani [4] modified the notion of fuzzy metric spaces with the help of t-norms in 1994. GÄahler [3] investigated 2-metric spaces in a series of his papers. Sharma, Sharma and Iseki [12] investigated, for the first time, contraction type mappings in 2-metric spaces. Many authors have studied common fixed point theorems in fuzzy metric spaces. Some of interesting papers are Y. J. Cho [1], George and Veeramani [4], Grabiec [5], Kramosil and Michalek [8] and S. Sharma [11]. S. H. Cho [2] proved a common fixed point theorem for four mappings in fuzzy metric spaces and S. Sharma [11] proved a common fixed point theorem for three mappings in fuzzy 2-metric spaces. In this paper we prove a common fixed point theorem for four mappings in fuzzy 2-metric spaces. Our theorem is an extension of results of S. H. Cho [2] to fuzzy 3-metric spaces. And also, it is a generalization of result of and S. Sharma [11].

Gahler in a series of papers [13, 14, and 15] investigated 2-metric spaces. Sharma, Sharma and Iseki [30] studied for the first time contraction type mappings in 2-metic space. We [34, 35] have also worked on 2-Metric spaces and 2- Banach spaces for rational expressions.

We know that 2 and 3-metric space is a real valued function of a point triples and four tuples respectively on a set X, which abstract properties were suggested by the area and volume function in Euclidean spaces. Fixed point theorems realted to fuzzy 2 and fuzzy 3 metric spaces were established by Shrivastav and Singhal [36]. Now it is natural to expect 4-Metric space for further investigation.

2. SOME FIXED POINT THEOREMS IN FUZZY 2-METRIC SPACE

Definition (2 A): A binary operation *: [0, 1] x [0,1] x [0,1] → [0,1] is called a continuous t-norm if ([0,1],*) is an abelian topological monodies with unit 1 such that $a_1 * b_1 * c_1 \geq a_2 * b_2 * c_2$ whenever $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$, for all a_1, a_2 , b_1, b_2,and c_1, c_2 are in [0,1].

Definition (2 B): The 3-tuple (X, M, *) is called a fuzzy 2-metric space if X is an arbitrary set, * is continuous t-norm and M is fuzzy set in $X^3 \times [0,\infty)$ satisfying the followings

$(FM'-1)$: $M(x,y,z,0) = 0$

$(FM'-2)$: $M(x,y,z,t) = 1, \forall\, t \succ 0, \Leftrightarrow x = y$

$(FM'-3)$: $M(x,y,t) = M(x,z,y,t) = M(y,z,x,t)$, Summary about three variable

$(FM'-4)$: $M(x,y,z,t_1,t_2,t_3) \geq M(x,y,u,t_1) * M(x,u,z,t_2) * M(x,y,z,t_3)$

$(FM'-5)$: $M(x,y,z)\colon [0,1) \to [0,1] is\ left\ continuous, \forall\, x,y,z,u\, \varepsilon\, X, t_1,t_2,t_3 \succ 0$

Definition (2 C): Let (X, M, *) be a fuzzy 2-matric space. A sequence $\{x_n\}$ in fuzzy 2-metric space X is said to be convergent to a point $x\varepsilon\, X$,

$\lim_{n\to\infty} M(x_n, x, a, t) = 1, for\ all\ a\varepsilon X\ and\ t \succ 0$

A sequence $\{x_n\}$ in fuzzy 2-metric space X is called a Cauchy sequence, if

$\lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1, for\ all\ a\varepsilon X\ and\ t, p \succ 0$

A fuzzy 2-matric space in which every Cauchy sequence is convergent is said to be complete.

Definition (2 D): A function M is continuous in fuzzy 2-metric space, iff whenever for all $a\varepsilon X$ and $t > 0$.

$x_n \to x, y_n \to y, then \lim_{n\to\infty} M(x_n, y_n, a, t) = M(x,y,a,t), \forall\, a\varepsilon X\ and\ t \succ 0$

Definition (2 E): Two mappings A and S on fuzzy 2-metric space X are weakly commuting iff

$M(ASu, SAu, a, t) \geq M(Au, Su, a, t), \forall u, a\ \varepsilon\ X\ and\ t \succ 0$

Theorem 2.1. Let (X, M, *) be a complete fuzzy 2-metric space. Let f and g be weakly compatible self maps of X satisfying

$M\left(g_x, g_y, a, k_t\right) \geq M\left(f_x, f_y, a, t\right) where\ \ 0 < k < 1, a > 0$

$g(X) \subseteq f(X)$

If one of $g(X) or\ f(X)$ is complete then f and g have a unique common fixed point.

Proof. Let $x_0 \varepsilon\, X$. Since $g(X) \subseteq f(X)$. Choose $x_1 \varepsilon\, X$ such that $g(x_0) = f(x_1)$. In general, choose x_{n+1} such that $y_n = fx_{n+1} = gx_n$. Then by (3.1), we have

$$M(fx_n, fx_{n+1}, a, t) = M(gx_{n-1}, gx_n, a, t) \geq M\left(fx_{n-1}, fx_n, a, \frac{t}{k}\right)$$

$$= M\left(gx_{n-2}, gx_{n-1}, a, \frac{t}{k}\right) \geq \cdots \ldots \ldots \geq M\left(fx_0, fx_0, a, \frac{t}{k^n}\right).$$

Therefore, for any p,

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$$M(fx_n, fx_{n+p}, a, t) \geq M\left(fx_n, fx_{n+1}, a, \frac{t}{p}\right) \geq \cdots . \geq M\left(fx_{n+p-1}, fx_{n+p}, a, \frac{t}{p}\right)$$

$$\geq M\left(fx_0, fx_1, a, \frac{t}{pk^n}\right) \geq \cdots . \geq M\left(fx_0, fx_1, a, \frac{t}{pk^{n+p-1}}\right)$$

As $n \to \infty$. $\{fx_n\} = \{y_n\}$ is a Cauchy sequence in fuzzy 2-metric space and so, by completeness of X, $\{y_n\} = \{fx_n\}$ is convergent. We call the limit z, then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$. As f(X) is complete, so there exist a point p in X such that $f_p = z$. Now, from (3.1),

As $n \to \infty$. $M(gp, gx_n, a, kt) \geq M(fp, fx_n, a, t)$,

$M(gp, z, a, kt) \geq M(fp, z, a, t)$,

$M(gp, z, a, kt) \geq M(z, z, a, t)$,

$M(gp, z, a, kt) \geq 1$,

$M(gp, z, a, kt) = 1$,

$gp = z = fp$.

As f and g are weakly compatible. Therefore fgp=gfp i.e. fz=gz. Now, we show that z is fixed point of f and g. From (3.1),

As $n \to \infty$. $M(gz, gx_n, a, kt) \geq M(fz, fx_n, a, t)$,

$M(gz, z, a, kt) \geq M(fz, z, a, t)$,

$M(gz, z, a, kt) \geq M(gz, z, a, t)$,

$gz = z = fz$.

Hence z is a common fixed point of f and g. For uniqueness, let w be another fixed point of f and g. Then by (3.1), $M(gz, gw, a, kt) \geq M(fz, fw, a, t), M(z, w, a, kt) \geq M(z, w, a, t)$ and z=w. Therefore z is unique common fixed point of f and g.

Theorem 2.2. Let (X, M, *) be a fuzzy 2-metric space. Let f and g weakly compatible self maps of X satisfying condition (3.1) and (3.2). If one of g(X) or f(X) is complete then f and g have a unique common fixed point.

Proof. From the proof of above theorem. We conclude that $\{fx_n\} = \{y_n\}$ is a Cauchy sequence in X. Now suppose that f(X) is a complete subspace of X. Then the subsequence of $\{y_n\}$ must get a limit in f (X). Call it be u and f(v) = u. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence, therefore the sequence $\{y_n\}$ also converges implying thereby the convergence of subsequence of the convergent sequence. Now, from (3.1),

As $n \to \infty$. $M(gv, gx_n, a, kt) \geq M(fv, fx_n, a, t)$,

$M(gv, u, a, kt) \geq M(fv, u, a, t)$,

$M(gv, u, a, kt) \geq M(u, u, a, t)$,

$M(gv, u, a, kt) \geq 1$,

$M(gv,u,a,kt) = 1,$

$gv = u = fv.$

Which shows that pair (f,g) has a point of coincidence. Since, f and g are weakly compatible, fgv = gfv, i.e. fu = gu. Now, we show that u is a fixed point of f and g. From (3.1).

As $n \to \infty$. $M(gu, gx_n, a, kt) \geq M(fu, fx_n, a, t),$

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Hence u is a fixed point of f and g. For uniqueness, let w be another fixed point of f and g. Then by (3.1), $M(gz, gw, a, kt) \geq M(fz, fw, a, t), M(z, w, a, kt) \geq M(z, w, a, t)$ and z=w.

Therefore z is unique common fixed point of f and g.

3. SOME FIXED POINT THEOREMS IN FUZZY 3-METRIC SPACES

Definition (3 A): A binary operation *: [0, 1] x [0,1] x [0,1] x [0,1] → [0,1] is called a continuous t-norm if ([0,1],*) is an abelian topological monodies with unit 1 such that $a_1 * b_1 * c_1 * d \geq a_2 * b_2 * c_2 * d_2$ whenever $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$ and $d_1 \geq d_2$for all a_1, a_2 , b_1, b_2, c_1, c_2 and $d_1, d_2$are in [0,1].

Definition (3 B): The 3-tuple (X, M, *) is called a fuzzy 3-metric space if X is an arbitrary set, * is continuous t-norm and M is fuzzy set in $X^4 \times [0, \infty)$ satisfying the followings

$(FM'' - 1)$: $M(x, y, z, w, 0) = 0$

$(FM'' - 2)$: $M(x, y, z, w, t) = 1, \forall\, t \succ 0$

$(FM'' - 3)$: $M(x, y, z, w, t) = M(x, w, z, y, t) = M(z, w, x, y, t) = \cdots$ $(FM'' - 4)$: $M(x, y, z, w, t_1 + t_2 + t_3) \geq M(x, y, z, u, t_1) * M(x, y, u, w, t_2) * M(x, u, z, w, t_3) * M(x, y, z, w, t_4)$

$(FM'' - 5)$: $M(x, y, z, w)$: $[0,1) \to [0,1]$ $is\ left\ continuous, \forall\, x, y, z, u\ \varepsilon\ X, t_1, t_2, t_3, t_4 \succ 0$

Definition (3 C): Let (X, M, *) be a fuzzy 3-matric space. A sequence $\{x_n\}$ in fuzzy 3-metric space X is said to be convergent to a point $x \varepsilon\ X$,

$\lim_{n\to\infty} M(x_n, x, a, b, t) = 1, for\ all\ a, b\ \varepsilon\ X\ and\ t \succ 0$

A sequence $\{x_n\}$ in fuzzy 3-metric space X is called a Cauchy sequence, if

$\lim_{n\to\infty} M(x_{n+p}, x_n, a, b, t) = 1, for\ all\ a, b\ \varepsilon\ X\ and\ t, p \succ 0$

A fuzzy 3-matric space in which every Cauchy sequence is convergent is said to be complete.

Definition (3 D): A function M is continuous in fuzzy 3-metric space, iff whenever for all $a \varepsilon X$ and $t > 0$.

$x_n \to x, y_n \to y, then \lim_{n\to\infty} M(x_n, y_n, a, b, t) = M(x, y, a, t), \forall\, a, b\ \varepsilon\ X\ and\ t \succ 0$

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Definition (3 E): Two mappings A and S on fuzzy 3-metric space X are weakly commuting iff $M(ASu, SAu, a, b, t) \geq M(Au, Su, a, t), \forall u, a, b\ \varepsilon\ X\ and\ t \succ 0$.

Theorem 3.1. Let (X, M, *) be a complete fuzzy 3-metric space. Let f and g be weakly compatible self maps of X satisfying

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$$M(fx_n, fx_{n+1}, a, b, t) = M(gx_{n-1}, gx_n, a, b, t) \geq M\left(fx_{n-1}, fx_n, a, b, \frac{t}{k}\right)$$
$$= M\left(gx_{n-2}, gx_{n-1}, a, b, \frac{t}{k}\right) \geq \cdots \ldots . \geq M\left(fx_0, fx_0, a, b, \frac{t}{k^n}\right).$$

Therefore, for any p,

$$M(fx_n, fx_{n+p}, a, b, t) \geq M\left(fx_n, fx_{n+1}, a, b, \frac{t}{p}\right) \geq \cdots . \geq M\left(fx_{n+p-1}, fx_{n+p}, a, b, \frac{t}{p}\right)$$
$$\geq M\left(fx_0, fx_1, a, b, \frac{t}{pk^n}\right) \geq \cdots . \geq M\left(fx_0, fx_1, a, b, \frac{t}{pk^{n+p-1}}\right)$$

As $n \to \infty$. $\{fx_n\} = \{y_n\}$ is a Cauchy sequence in fuzzy 3-metric space and so, by completeness of X, $\{y_n\} = \{fx_n\}$ is convergent. We call the limit z, then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$. As f(X) is complete, so there exist a point p in X such that $f_p = z$. Now, from (3.1),

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Hence z is a common fixed point of f and g. For uniqueness, let w be another fixed point of f and g. Then by (3.1), $M(gz, gw, a, b, kt) \geq M(fz, fw, a, b, t), M(z, w, a, b, kt) \geq M(z, w, a, b, t)$ and z=w.

Therefore z is unique common fixed point of f and g.

Theorem 3.2. Let (X, M, *) be a fuzzy 3-metric space. Let f and g weakly compatible self maps of X satisfying condition (4.1) and (4.2). If one of g(X) or f(X) is complete then f and g have a unique common fixed point.

Proof From the proof of above theorem. We conclude that $\{fx_n\} = \{y_n\}$ is a Cauchy sequence in X. Now suppose that f(X) is a complete subspace of X. Then the subsequence of $\{y_n\}$ must get a limit in f (X). Call it be u and f(v) = u. As $\{y_n\}$ is a Cauchy sequence containing a convergent subsequence, therefore the sequence $\{y_n\}$ also converges implying thereby the convergence of subsequence of the convergent sequence. Now, from (4.1),

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$M(gv, u, a, b, kt) \geq M(fv, u, a, b, t)$,

$M(gv, u, a, b, kt) \geq M(u, u, a, b, t)$,

$M(gv, u, a, b, kt) \geq 1$,

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Hence u is a fixed point of f and g. For uniqueness, let w be another fixed point of f and g. Then by (3.1), $M(gz, gw, a, b, kt) \geq M(fz, fw, a, b, t), M(z, w, a, b, kt) \geq M(z, w, a, b, t)$ and z=w.

Therefore z is unique common fixed point of f and g.

4. SOME FIXED POINT THEOREMS IN FUZZY 4-METRIC SPACES

Definition (4 A): A binary operation *: $[0,1]^5 \to [0,1]$ is called a continous t-norm if $\{[0,1],*\}$is an abelian topological monodies with unit 1 such that $a_1 * b_1 * c_1 * d_1 * e_1 \geq a_2 * b_2 * c_2 * d_2 * e_2$ whenever $a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2, d_1 \geq d_2$ and $e_1 \geq e_2$ for all $a_1, a_2, b_1 b_2, c_1, c_2, d_1, d_2$ and e_1, e_2 are in [0,1].

Definition (4 B): The 3-tuple $(X, M,*)$ is called a fuzzy 4-metric space if X is an arbitrary set, * is continuous t-norm monoid and M is a fuzzy set in $X^4 \times [0, \infty]$ satisfying the following conditions:

(1) $M(x, y, z, v, w, 0) = 0$
(2) $M(x, y, z, v, w, t) = 1 \forall t > 0$
(3) $M(x, y, z, v, w, t) = M(x, v, u, y, w, t) = M(u, y, x, v, w, t) = \cdots \dots \dots \dots ..$
(4) $M(x, y, z, v, w, t_1 + t_2 + t_3 + t_4 + t_5) \geq M(x, y, u, v, w, t_1) + M(x, y, u, v, w, t_2) + M(x, y, z, v, w, t_3) + M(x, y, u, v, w, t_4) + M(x, y, u, v, w, t_5)$
(5) $M(x, y, z, v, w)$: [0,1] → [0,1] is left continous, $\forall x, y, u, v, w \in X, t_1, t_2, t_3, t_4, t_5 > 0$

Definition (4 C): Let $(X, M,*)$ be a fuzzy 4 metric space:

(1) A sequence $\{x_n\}$ in fuzzy 4 metric space X is called convergent to appoint $x \in X$, if $\lim_{n\to\infty} M(x_n, x, a, b, c, t) = 1$, for all $a, b \in X$ and $t > 0$
(2) A sequence $\{x_n\}$ in fuzzy 4 metric space X is called Cauchy sequence, if $\lim_{n\to\infty} M\left(x_{n+p}, x, a, b, c, t\right) = 1$, for all $a, b \in X$ and $t, p > 0$
(3) A fuzzy 4 metric space in which every Cauchy sequence is convergent is said to be complete.

Definition (4 D): A function M is continuous in fuzzy 3-metric space if $x_n \to x, y_n \to y$, then $\lim_{n\to\infty} M(x_n, y_n, a, b, c, t) = M(x, y, a, b, c, t), \forall\, a, b \in X$ and $t > 0$

Definition (4 E): Two mappings A and S on fuzzy 4-metric space X are weakly commuting iff, $M(ASu, SAu, a, b, c, t) \geq M(Au, Su, a, b, c, t) \forall\, u, a, b, c \in X$ and $t > 0$

Theorem (4.1) Let $(X, M,*)$ be a complete fuzzy 4-metric space. Let f and g be weakly compatible self maps of X satisfying

(5.1) $M(gx, gy, a, b, c, kt) \geq M(fx, fy, a, b, c, t)$ where $0 < k < 1, a, b > 0$

\ (5.2) $g(X) \subseteq f(X)$

If one of $g(X)$ or $f(X)$ is complete then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. Since $g(X) \subseteq f(X)$. Choose $x_1 \in X$ such that $g(x_0) = f(x_1)$. In general, choose x_{n+1} such that $y_n = fx_{n+1} = gx_n$. Then, we have

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$$M(fx_n, fx_{n+1}, a, b, c, t) = M(gx_{n-1,}gx_n, a, b, c, t) \geq M\left(fx_{n-1}, fx_n, a, b, c, \frac{t}{k}\right)$$

$$= M\left(gx_{n-2,}gx_{n-1}, a, b, c, \frac{t}{k}\right) \geq \cdots \ldots \ldots \ldots \ldots . \geq M\left(fx_0, fx_1, a, b, c, \frac{t}{k^n}\right)$$

Therefore for any p,

$$M(fx_n, fx_{n+p}, a, b, c, t) \geq M\left(fx_n, fx_{n+1}, a, b, c, \frac{t}{p}\right) \ldots \ldots \ldots \ldots \geq M\left(fx_{n+p-1}, fx_{n+p}, a, b, c, \frac{t}{p}\right)$$

$$\geq M\left(fx_0, fx_1, a, b, c, \frac{t}{pk^n}\right) \geq \cdots \ldots \ldots \ldots \geq M\left(fx_0, fx_1, a, b, c, \frac{t}{pk^{n+p-1}}\right)$$

As $n \to \infty$. $\{fx_n\} = \{y_n\}$ is a Cauchy sequence in fuzzy 4-metric space and so, by completeness of X, $\{y_n\} = \{fx_n\}$ is convergent. We call the limit z, then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$. As $f(X)$ is complete, so there exist a point in X such that $fp = z$. Now,

As $n \to \infty$, $M(gp, gx_n, a, b, c, kt) \geq M(fp, fx_n, a, b, c, t)$

$M(gp, z, a, b, c, kt) \geq M(fp, z, a, b, c, t)$

$M(gp, z, a, b, c, kt) \geq M(z, z, a, b, c, t)$

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$M(gz, z, a, b, c, kt) \geq M(gz, z, a, b, c, t)$

$gz = z = fz.$

Hence z is a common fixed point of f and g. For uniqueness, let q be another fixed point of f and g. Then

$M(gz, gq, a, b, c, kt) \geq M(fz, fq, a, b, c, t), M(z, q, a, b, c, kt) \geq M(z, q, a, b, c, t)$ and z=q.

Therefore z is unique common fixed point of f and g.

Theorem (4.2) Let $(X, M, *)$ be a fuzzy 4-metric space. Let f and g are weakly compatible self maps of X satisfying conditions (5.1) and (5.2).

Proof. From the proof of above theorem. We conclude that $\{fx_n\} = \{y_n\}$ is a Cauchy sequence in X. Now suppose that $f(X)$ is complete subspace of X. Then the subsequence of $\{y_n\}$ must get a limit in $f(X)$. Call it be r and f(s)=r. Then $\{y_n\}$ is a Cauchy sequence containing a subsequence, therefore the sequence also convergent implying thereby the convergence of the convergent sequence. Now, from (5.1),

As $n \to \infty$, $M(gs, gx_n, a, b, c, kt) \geq M(fs, fx_n, a, b, c, t)$,

$M(gs, r, a, b, c, kt) \geq M(fs, r, a, b, c, t)$,

$M(gs, r, a, b, c, kt) \geq M(r, r, a, b, c, t)$,

$M(gs, r, a, b, c, kt) \geq 1$,

$M(gs, r, a, b, c, kt) = 1$,

$gs = r = fs$

Which shows that pair (f, g) have a point of coincidence. Since, f and g are weakly compatible, $fgs = gfs, i.e. fr = gr$.

Now, we show that r is a fixed point of f and g. From (5.1).

$M(gr, gx_n, a, b, c, kt) \geq M(fr, fx_n, a, b, c, t)$,

$M(gr, r, a, b, c, kt) \geq M(fr, u, a, b, c, t)$,

$M(gr, gx_n, a, b, c, kt) \geq M(gr, fx_n, a, b, c, t)$,

$gr = r = fr$

Hence r is a fixed point of f and g. For uniqueness, let d be be another fixed point of f and g. Then by (4.1).

$M(gr, gd, a, b, c, kt) \geq M(fr, fd, a, b, c, t)$,

$M(r, d, a, b, c, kt) \geq M(r, d, a, b, c, t)$, and
$r = d$. Therefore r is unique common fixed point of f and g.

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