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Effect of Impulse on Modeling Price Fluctuations in Single Commodity Markets

KEYWORDS: Impulsive functional differential equation, uniformly asymptotically stable, second method of Lyapunov.

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ABSTRACT *We investigate effect of impulse on the model of price fluctuation in single commodity market. We established that impulsive functional differential equation is more adequate as a modeling tool. We show the roles of impulses in changing the behavior pattern of solution of impulsive functional differential equation. Using second method of Lyapunov , we are able to establish the stability of the model.*

1. INTRODUCTION :

In a single good market, there are three variables: the quantity demanded q_d , the quantity supplied q_s and the price p . The equilibrium is attained when the excess demand is zero $q_d - q_s = 0$, that is the market is cleared. But generally the market is not in equilibrium and at an initial time t_0 the price p_0 is not at the equilibrium value $\bar{p}$, that is $p_0 \neq \bar{p}$. In such a situation the variables q_d , q_s *and* p must change over time and are considered as function of time. The dynamic question is given sufficient time, how as the adjustment process $p(t) \rightarrow \bar{p}$ as $t \rightarrow \infty$ to be described?

The dynamic process of attaining equilibrium in a single market model is tentatively described by differential equations, on the basis of considerations on price changes governing the relative strength of the demand and supply forces. For the sake of simplicity, the rate of price change with respect to time is assumed to be proportional to excess demand $q_d - q_s$. Moreover, the definitive relationships between the market price p of a commodity, the quantity demanded and the quantity supplied are assumed to exist. These relationships are called the demand and supply curve, occasionally modeled by demand function $q_d = q_d(p)$ or a supply function $q_s = q_s(p)$. In a case where the rate of price change with respect to time is assumed to be proportional to the excess demand, the differential equation belongs to the class

$$\frac{1}{p}\frac{dp}{dt} = f\left(q_s(p), q_d(p)\right), \quad \ldots\ldots\ldots(1.1)$$

of differential equations . The question that arises is about the nature of the time path, resulting from equation (1.1). In [9] Muresan studied special a case of fluctuation model with time delay of the form

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$$\frac{dp}{dt} = \left(\frac{a}{b + p^q(t)} - \frac{c\,p^r\big(g(t)\big)}{d + p^r\big(g(t)\big)}\right)p(t) \quad \ldots\ldots\ldots (1.2)$$

$where\ a, b, c, d, r > 0, q \in [1, \infty),\ g \in C[R_+, R_+]$, and proved that there exist a positive bounded unique solution.

Rus and Iancu [10] generalized the model equation (1.2) and studied the model of the form

$$\begin{bmatrix} \dfrac{dp}{dt} = F\big(p(t)\big), p(t-\tau))p(t), \quad t \geq 0 \\ p(t) = \varphi(t), \quad t \in [-\tau, 0] \end{bmatrix} \quad \ldots . (1.3)$$

They proved the existence and uniqueness of the equilibrium solution of the model considered and established some relations between this solution and coincidence points.

2. IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS : An empirical time series analysis [4,12] of German macroeconomic data emphasized to model capital intensity subject to short term perturbations at certain moments of time. It is unlikely to have a regular solution of equation (1.3). The solution must have some jumps that follows a regular pattern. An adequate mathematical model for a long term planning will be the following impulsive functional differential equation

$$\begin{bmatrix} \dfrac{dp}{dt} = F(p(t), p_t)p(t), \quad t \geq 0,\ t \neq t_i \\ \Delta p(t_i) = p(t_i + 0) - p(t_i) = p_i(p(t_i),\ i = 1,2, \ldots . \end{bmatrix} \quad \ldots . (2.1)$$

Where $t_0 \in R_+;\ t_0 < t_1 < t_2 < \cdots \ldots,\ \lim_{i \to \infty} t_i = \infty;\ \Omega\ be\ a\ domain\ in\ R_+$, Containing the origin ; $F : \Omega\ X\ PC\{[-\tau, 0], \Omega\} \to R\ ;\ p_i : \Omega \to R, i = 1,2, \cdots\cdots$ are functions that characterize the magnitude of the impulsive effect at the time t_i. $p(t_i)\ and\ p(t_i + 0)$ are respectively the price levels before and after the impulse effects and for $t \geq t_0,\ p_t \in PC\{[-\tau, 0], \Omega\}\ is\ desigined\ by\ p_t(s) = p(t + s),\quad -\tau \leq s \leq 0$

$Let\ p_0 \in CB\{[-\tau, 0], \Omega\}, \quad p(t) = p(t; t_0, p_0), p \in \Omega, the\ solution\ of\ the\ equation\ (2.1)$

Satisfying the initial conditions
$$\begin{bmatrix} p(t; t_0, p_0) = p_0(t - t_0), \quad t - \tau \leq t \leq t_0 \\ p(t_0 + 0, t_0, p_0) = p_0(0) \end{bmatrix} \quad \ldots . (2.2)$$

$J^+(t_0, p_0)\ the\ maximal\ interval\ of\ type\ [t_0, \beta)\ in\ which\ the\ solution\ p(t; t_0, p_0)$ is define and by $||p_0||_\tau = \max_{t \in [t_{0-\tau}, t_0]} ||p_0(t - t_0)||$ the norm of the function $p_0 \in CB\{[-\tau, 0], \Omega\}$

Let us assume the following conditions :

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$H_1 \;-\; The\ function\ F\ is\ continuous\ on\ \Omega\ X\ PC\{[-\tau,0],\Omega\}$

$H_2 \;-\; The\ function\ \ F\ is\ locally\ \ lipschitz\ continuos\ \ with\ respect\ to\ its\ second\ argument\ on$

$\Omega\ X\ PC\{[-\tau,0],\Omega\}$

$H_3 \;-\; There\ exist\ a\ constant\ m > 0\ such\ that\ |F(p,p_t)| \leq\ m\ < \infty\ \ for\ \ (p,p_t) \in\ \Omega\ X\ PC\{[-\tau,0],\Omega\}$

$H_4 \;-\; p_i\ \in [\,\Omega, R],\ \ i = 1,2,\cdots\cdots$

$H_5 \;-\; The\ functions\ (I + p_i):\ \ \Omega \to\ \Omega\ ,\ \ i = 1,2,\cdots\cdots, where\ i\ is\ the\ identity\ in\ \ \Omega$

$H_6 \;-\; t_0 <\ t_1\ <\ t_2 < \cdots \ldots,\ \ \lim_{i\to\infty} t_i = \infty$

$H_7 -\ \ Let\ there\ exist\ a\ piecewise\ continuous\ function\ \ \ V : [\ t_0, \infty)\ X\ \Omega\ \to\ R_+\ which\ belong\ to$

$class\ \ V_0\ and\ for\ which\ \ V\big(t, p^*(t)\big) = 0\ , t \geq\ t_0$

Under the assumption of hypothesis H_6 we define

$G_k = [(t,P) \colon \tau_{k-1}\,(p)\ < t <\ \tau_k\,(p)\ ,\ \ \ p\ \in\ \Omega,\ \ \ \ k = 1,2,\ldots,$ ....(2.3)

$Let\ p_i\ \in CB\{[-\tau,0],\Omega\}\ ,\ \ \ \ \ \ p^*(t) =\ \ p^*(t,t_0,p_i)\ , p^*\ \in\ \Omega\ \ is\ the\ solution\ of\ \ equation\ (2.1)$

Satisfying the initial conditions $\begin{bmatrix} p^*(t;t_0,p_1) =\ p_1(t-t_0), \quad t_0 - \tau\ \leq t \leq\ t_0 \\ p^*(t_0+0,t_0,p_1) =\ p_1(0) \end{bmatrix}$ ....(2.4)

For a function $V\ \in\ V_0\ , and\ some\ , t \geq\ t_0$ we shall use also the class

$\Omega_1 = [p \in PC[\,t_0,\infty),\Omega] \colon V\big(s,p(s)\big) \leq V\big(t,p(t)\big),\ \ \ t - \tau\ \leq s \leq\ t\,]$

REMARK : Corollary 1 and theorem (1) are going to be useful in establishing the stability of our model equation

<u>Corollary 1</u> : Assume that (i) condition $H_1 - H_6\ holds$

(ii) The function $V\ \in\ V_0\ is\ such\ that\ \ V\big(t+0, p +\ I_k(p)\big) \leq V(t,p)\ \ ,\ \ p\ \in\ \Omega, t =\ t_k\ \ \ k = 1,2,\ldots$

and the inequality $D_{2.5}\,\big(V,t,\ p(t)\big) \leq 0\ , t\ \neq\ t_k\ , k = 1,2,\cdots\cdots, is\ valid\ for\ t\ \in [t_0\,,\infty), p \in\ \Omega_p$

Then $V\big(t,p(t,t_0,\varphi_0)\big) \leq V\big(t_0 + 0, \varphi_0(0)\big),\ \ t\ \in [t_0\,,\infty)$

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Theorem 1 : *Assume that (i) condition* $H_1 - H_6$ *holds*

(ii) The function G : $[t_0, \infty) \, X \, R_+ \to R$ *is continuous in each of the sets* $[t_{k-1}, t] \, X \, R_+$, $k = 1,2, \ldots$

(iii)$B_k \in C[R_+, R_+]$ *and* $\varphi_k(u) = u + B_k(u) \geq 0, k = 1,2, \ldots$ *are nondecreasing with respect to u*

(iv) The maximal solution of the problem

$$\begin{cases} \frac{du}{dt} = G\big(t, u(t)\big), t \neq t_k \\ u(t_0) = u_0 \geq 0 \\ \Delta u(t_k) = B_k(u(t_k), \; t_k > t_0, \; k = 1,2, \ldots \end{cases} \quad \ldots.(2.5)$$

is defined in the interval $[t_0, \infty)$,

(*v*) *The function* $V \in V_0$ *is such that* $V\big(t_0 + 0, \varphi_0(0)\big) \leq u_0$,

$$V\big(t_0 + 0, p, I_k(p)\big) \leq \varphi_k(V(t,p), \qquad p \in \Omega, t = t_k \quad k = 1,2, \ldots$$

and the inequality

$$D^+_{2.5} V\big(t, x(t)\big) \leq \varphi_k(V(t,x), x \in \Omega, \; t = t_k, k = 1,2, \cdots\cdots, \text{is valid for } t \in [t_0, \infty), \in \Omega_x$$

Then $V\big(t, x(t, t_0, \varphi_0)\big) \leq u^+(t, \; t_0, \; u_0), \; t \in [t_0, \infty)$

Theorem 2 : *Assume that condition (i)* $H_1 - H_6$ *holds*

(2) There exist a function $V \in V_0$ *such that* H_7 *holds*

$a(|p - p^*(t)| \leq V(t,p), \; (t,p) \in [t_0, \infty) \, X \, \Omega, a \in k$ *.....(2.6)*

$V\big(t + 0, p + I_k(p)\big) \leq V(t,p), p \in \Omega$ *, then the solution* $p^*(t)$ *of equation* (1.4) *is stable.*

Proof :

Let $\varepsilon > 0$, it follows from the properties of the function V that $\exists$ a constant $\delta = \delta(t_0, \varepsilon) > 0$

such that if $x \in \Omega$. $\|x\| < \delta$, then $sup_{||x||<\delta} V(t_0 + 0, x) < a(\varepsilon)$

Let $\varphi_0 \in PC\{[-r, 0], \Omega]$: $||\varphi_0||_r < \delta$, then $\big\|\varphi_p(0)\big\| \leq ||\varphi_0||_r < \delta$

$$\text{and} \qquad V\big(t_0 + 0, \varphi_0(0)\big) < a(\varepsilon) \qquad \ldots\ldots..(2.7)$$

Let $p(t) = p^*(t; \, t_0, \varphi_0)$ be the solution of problem (2.1), since all the conditions of corollary 1

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are met, then $V(t, x(t; t_0, \varphi_0) \leq V\big(t_0 + 0, \varphi_0(0)\big), t \in [t_0, \infty)$ ……… (2.8)

From (2.6), (2.7) and (2.8) there follows the inequality

$$a(|p - p^*(t)| \leq V(t, p(t, t_0, \varphi_0) \leq V\big(t_0 + 0, \varphi_0(0)\big) < V(t, p)$$

Whence we obtain that $|p^*(t;\ t_0, \varphi_0)| < \varepsilon\ for\ t \geq t_0$. This implies that solution $p^*(t)$ of equation (2.1) is stable ■

Theorem 3 : Let the condition of theorem (2) hold and let a function $b \in k$ such that

$V(t, P) \leq b(|p - p^*(t)|),\quad (t, p) \in [t_0, \infty)\ X\ \Omega$ ……… (2.9), then the solution $p^*(t)$ *of problem* (2.1) is uniformly stable.

Proof : Let $\varepsilon > 0$ *be* given choose $\delta = \delta(\varepsilon) > 0$ so that $b(\delta) < a(\varepsilon)$.

Let $\varphi_0 \in PC\{[-r, 0], \Omega\}$: $\left||\varphi_0|\right|_r < \delta\ and\ p^*(t) = p^*(t;\ t_0, \varphi_0)$ be the solution of problem (2.1) and (2.2). As in theorem 2, we prove that

$$a(|p\ (t; t_0, \varphi_0, |) \leq V(t, p(t, t_0, \varphi_0) \leq V\big(t_0 + 0, \varphi_0(0)\big), \qquad t \geq t_0 \quad \cdots\cdots (2.10)$$

From (2.9) and (2.10) we get to the inequalities

$$a(|p\ (t; t_0, \varphi_0, |) \leq V\big(t_0 + 0, \varphi_0(0)\big) \leq b|\varphi_0(0)|) \leq b(\left||\varphi_0|\right|_r) < b(\delta) < a(\varepsilon)$$

From which it follows that $|p\ (t; t_0, \varphi_0, | < \varepsilon\ , for\ \ t \geq t_0$ ■

Theorem 4 : Assume that

(i) Conditions $H_1 - H_6$ *holds*
(ii) There exist a function $V \in V_0$ *such that* H_7 *holds*

$$a(|p - p^*(t)| \leq V(t, p) \leq b(|p - p^*(t)|,\ \ (t, p) \in [t_0, \infty)\ X\ \Omega, a, b \in k,\ p \in \Omega, t = t_i, i = 1,2, ..$$

$$V\big(t + 0, p + I_k(p)\big) \leq V(t, p) \quad \ldots\ldots(2.11)$$

$$and\ the\ inequality\ D^+_{(2.1)}V\big(t, p(t)\big) \leq -c\ (|p(t) - p^*(t)|), \qquad t \neq t_i, \qquad i = 1,2,.., \quad (2.12)$$

Then the solution $p^*(t)$ *of problem* (2.1) is uniformly asymtotically stable.

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Proof : 1 . Let $\alpha = constant\ > 0$, $[\, p \in R^n\ , \{|p - p^*(t)| \leq \alpha\} \complement\ \Omega$.

For any $t\ \in [t_0\,, \infty)\ denote\ V_{t,\alpha}^{-1}\ = [p \in\ \Omega\ : V(t+0,p) \leq a(\alpha)$

$$from\ (2.8), we\ deduce\ V_{t,\alpha}^{-1}\ \complement\ [p \in\ R^n :\ |p - p^*(t)| \leq\ \alpha]\ \complement\ \Omega\ .$$

From condition (2) of theorem 4 , it follows that for any $t_0\ \in R\ \ and\ function\ \varphi_0 \in PC\ \{[-r,0],\Omega\}$:

$$\varphi_0(0) \in\ V_{t_0,\alpha}^{-1}\,, we\ have\ p(t;t_0,\varphi_0) \in\ V_{t,\alpha}^{-1}\,,\ t \geq\ t_0$$

Let $\varepsilon > 0$ be chosen. Choose $\eta = \eta(\varepsilon)$ so that $b(\eta) < a(\varepsilon)$ and let $t > \frac{b(a)}{C(\varepsilon)}$. If we assume that for each $t \in [t_0, t_0+T], the\ inequality\ |\, p(t;t_0,\varphi_0)| \geq \eta$ is valid then from (2.6) and (2.12) we get

$V(t,p(t;t_0,\varphi_0) \leq\ V\big(\,t_0+0,\varphi_0(0)\big) - \int_{t_0}^{t} c|\,p(s;t_0,\varphi_0)|)ds\ \leq b(\alpha) - c(\eta)T < 0$ which contradict (2.11).The contradiction obtained shows that there exist $t^* \in [t_0, t_0+T]\ such\ that\ |\, p(t^*;t_0,\varphi_0)| < \eta$

Then from (2.6) and (2.11) it follows that $t \geq t^*\ \ (hence\ for\ any\ t \geq\ t_0 + T\ , then\ the\ following\ inequality\ hold.\ \ a(|\, p(t;t_0,\varphi_0)|) < \varepsilon\ for$

$$t \geq\ t_0 + T.$$

2 Let $\lambda = constant > 0\ be\ such\ that\ b(\lambda) < a(\alpha).\ Then\ if\ \lambda = constant > 0\ be\ such\ that$

$$b(\lambda) < a(\alpha).\ Then\ if\ \varphi_0 \in PC\ \{[-r,0],\Omega\}\text{:}\ \ (||\varphi_0\,||)\_r < \delta :\ \ \|\varphi_0\|_r < \delta$$

(2.11) implies $V\big(\,t_0+0,\varphi_0(0)\big) \leq b(||\ \varphi_0(0)||) \leq b\big(\|\varphi_0\|_r\big) <\ b(\lambda) < a(\alpha)$ which shows that

$\varphi_0 \in PC\ \{[-r,0],\Omega\}$: $\ \varphi_0(0) \in\ V_{t_0,\alpha}^{-1}$ .From what we proved in item 1, it follows that the $p^*(t)\ \ of\ problem$ (2.1) is uniformly attractive and since theorem 3 implies uniform stability, then the solution $p^*(t)\ is\ uniformly$ asymtotically stable ■

Examples :

1. Let $a, b, c\ and\ d\ > 0, a\ linear\ demand\ function\ q_d = a - b_p\ and\ a\ linear\ supply$
   $function\ q_s =\ -c + d_p\ be\ given\ and\ the\ function\ f =\ \ \alpha(\ q_d -\ \ q_s),\qquad \alpha > 0$
   They can be put into equation (1.1), given the linear non-homogenous differential equation
   $\frac{dp}{dt} =\ \alpha(a+c-p(b+d) = p\alpha\left(\frac{a+c}{p} - (b+d)\right)$, corresponding to a the special type of differential equation (1.1). It complementary and particular solutions are immediate.

2. Let $\frac{dp}{dt}$ = $\alpha\left(\frac{a+c}{p(t)} - b - d\ \frac{p(t-\sigma(t))}{p(t)}\right)p(t)$ ……(2.13)
   Be a special case of model studied by Markey and Blair [151] where $0 \leq\ \sigma(t) \leq \tau,$
   $and\ \tau\ is\ a\ constant\ .If\ at\ the\ moments\ \ t_1, t_2, \cdots \cdots$

$(t_0 < t_1 < t_2 < \cdots \ldots < t_i < t_{i+1} < \cdots \ldots.\ and\ \lim_{i\to\infty} i = \infty)$ the above equation is subject to impulsive perturbations then the adequate model mathematical model is the following impulsive equation

$$\begin{cases} \frac{dp}{dt} = \alpha\left[\frac{a+c}{p(t)} - b - d\,\frac{p(t-\sigma(t))}{p(t)}\right]p(t) \\ \Delta p(t_i) = -\,\delta_i\left(p(t_i) - \frac{a+c}{b+d}\right), \qquad i = 1,,\ldots \end{cases} \quad \ldots..(2.14)$$

where $t_0 \in R_+, p(t)\ represent\ the\ price\ at\ moment\ t,\ \delta_i \in \mathrm{R}$ are constants. i = 1,2, … ..

$obviously\ p^* = \frac{a+c}{b+d}$ is an equilibrium of (2.14). Let there exist a constant $\beta > 0\ such\ that\ d \le b - \beta$

and the inequalities $0 < \delta_i <$ are valid for i=1,2,…., then p^* is uniformly asymptotically stable.

Given

$V(t,p) = \frac{1}{2}(p - p^*)^2 .then\ the\ set\ \Omega_1 = [p \in PC\big[[t_0,\infty),(0,\infty)\big]: (p(s) - p^*)^2 \le (p(t) - p^*)^2$

$$t - \tau \le s \le t, for\ t \ge t_0 , t \ne t_i, \qquad we\ have$$

$$D^+_{(2.14)} V\big(t,p(t)\big) = \alpha(p(t) - p^*)[a - bp(t) + c - dp(t - \sigma(t))]$$

Since p^* is an equilibrium of (2.14), we have

$$D^+_{(2.14)} V\big(t,p(t)\big) = \alpha(p(t) - p^*)[-bp(t) - p^*) - dp\big(t - \sigma(t)\big) - p^*]$$

From the last relation for $t \ge t_0 , t \ne t_i\ and\ p \in \Omega_1, we\ obtain\ the\ estimate$

$D^+_{(2.14)} V\big(t,p(t)\big) \le [-b + d]\,(p - p^*)^2 \le -\alpha\beta(p - p^*)^2 , also\ if\ 0 < \delta_1 < for\ all\ i = 1,2, \ldots, then$

$$V\big((t_i + 0), p(t_i + 0)\big) = \frac{1}{2}\,[(1 - \delta_i)p(t_i) + \delta_i p^* - p^*]^2 < V\big(t_i, p(t_i)\big).$$

Since all conditions of theorem 4 are satisfied p^* is uniformly asymptotically stable.

4 <u>CONCLUSION</u> : From the example given in section 3, we observe that if the constant δ_i are such that $\delta_i < 0\ or\ \delta_i > 2$, then condition 4 of theorem 3 is not satisfied and we cannot make any conclusion about asymptotic stability of p^*. This example demonstrate the utility of second method of Lyapunov. The main characteristic of the method is the introduction of a function, namely Lyapunov function which defines a generalized distance between p(t) and p^*.

By means of piecewise continuous function we give the conditions for uniform asymptotic of p^*. A technique is appllied, based on certain minimal subset.

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