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A Note on Properties of Maximum Likelihood Estimators for One-Way Repeated Measurements Model

KEYWORDS: one-way- RMM, ANOVA, maximum likelihood method, ratio test

Abdul-Hussein Saber Al-Mouel
Department of mathematics,College of Education for Pure Sciences,University of Basrah ,Basrah,Iraq

Hassan Raheem Showel Al-Shmailawi
Department of mathematics,College of Education for Pure Sciences,University of Basrah ,Basrah,Iraq

ABSTRACT *In this research, we consider the one- way repeated measurements model, (one-way- RMM), which has only one within units factor and one between units factor incorporating univariate random effects as well as the experimental error term,and investigate the analysis of variance (ANOVA) for the model. Also estimated the parameters of the model by using the maximum likelihood method as well as investigate some properties of estimators,and identify the analytic form of the likelihood-ratio test for the model.*

1. Introduction

Repeated measurements is a term used to describe data in which the response variable for each experimental units is observed on multiple occasions and possible under different experimental conditions [1]. Repeated measures is a common data structure with multiple measurements on a single unit repeated over time. Multivariate linear models with correlated errors. Repeated measurements analysis is widely used in many fields, for example , in the health and life science, epidemiology, biomedical, agricultural, psychological, educational researches and so on[1]. Repeated measurements occur frequently in observational studies which are longitudinal in nature,and in experimental studies incorporating repeated measures designs. [8].
Vonesh and Chinchilli (1997) discussed linear and nonlinear models for the analysis of repeated measurements [8].
Piterbarg And Rozovskii(1997) studied the problem of estimating parameters of randomly perturbed PDE's, the maximum likelihood estimators based on discrete time sampling of M observable Fourier coefficients of the random field governed by the stochastic PDE [7].
Al-Mouel (2004) studied the multivariate repeated measures models and comparison of estimators[1].
Al-Mouel and Wang (2004) presented the sphericity test for one –way multivariate repeated measurements analysis of variance model.They studied the asymptotic expansion of the sphericity test for one –way multivariate repeated measurements analysis of variance model[2].
Mohaisen and Swadi (2014) studied the Bayesian Estimaters of the one-way repeated measurements model[3].
In this research, in section two we consider the one- way repeated measurements model, (one-way- RMM), which has only one within units factor and one between units factor incorporating univariate random effects as well as the experimental error term,in section three investigates estimation of the parameters of the model by using maximum likelihood method , in section four investigates some properties of estimators, and identify the analytic form of the likelihood-ratio test for the repeated measurement model.

2 . One- way repeated measurements model

For convenience we concider the following linear model and parametrization for the one- way RMM with one between units factor and one within units factor incorporating univariate random effects .

$$y_{ijk} = \mu + \tau_j + \delta_{i(j)} + \gamma_k + e_{ijk} \qquad (2.1)$$

Volume : 6 | Issue : 9 | September 2016 | ISSN - 2249-555X | IF : 3.919 | IC Value : 74.50

where

i= 1,2,……,n is an index for experimental unit with group j,

j=1,2,……,q is an index for levels of the between-units factor (Group),

k=1,2,……,p is an index for levels of the within- units factor (Time),

y_{ijk} is the respons measurement at time k for unit i within group j ,

μ is the overall mean ,

τ_j is the added effect for treatment group j ,

$\delta_{i(j)}$ is the random effect for due to experimental unit i within treatment group j ,

Υ_k is the added effect for time k ,

e_{ijk} is the random error on time k for unit i within group j ,

For the parameterization to be of full rank, we imposed the following set of conditions

$$\sum_{j=1}^{q} \tau_j = 0 \quad , \quad \sum_{k=1}^{p} \Upsilon_k = 0$$

And we assume that e_{ijk}'s and $\delta_{i(j)}$'s are independent with

$e_{ijk} \sim$ i.i.d $N(0, \sigma_e^2)$, $\delta_{i(j)} \sim$ i.i.d $N(0, \sigma_\delta^2)$, (2.2)

Now we state the analysis of variance table for the one- way repeated measurements model including sum of square terms for group (SS_G), unit (group) ($SS_{U(G)}$),Time (SS_T), and error (SS_E),also the degree of freedom (d.f),mean square (MS),and expectation of mean square E(MS) as following:

Source of variance	d . f	SS	MS	E (MS)
Group	q-1	SS_G	$\frac{SS_G}{q-1}$	$\frac{np}{q-1}\sum_{j=1}^{q} \tau_j^2 + p\sigma_\delta^2 + \sigma_e^2$
Unit (Group)	q(n-1)	$SS_{U(G)}$	$\frac{SS_{U(G)}}{(n-1)\,(q-1)}$	$p\sigma_\delta^2 + \sigma_e^2$
Time	p-1	SS_{TIME}	$\frac{SS_{time}}{p-1}$	$\frac{nq}{p-1}\sum_{k=1}^{p} \Upsilon_k^2 + \sigma_e^2$
Residual	(p-1)(nq-1)	SS_E	$\frac{SS_E}{(p-1)(nq-1)}$	σ_e^2

ANOVA table for the one- way RMM.

Where:

$SS_G = np \sum_{j=1}^{q} (\bar{y}_{.j.} - \bar{y}_{...})^2$,

Volume : 6 | Issue : 9 | September 2016 | ISSN - 2249-555X | IF : 3.919 | IC Value : 74.50

$SS_{U(G)} = p\sum_{i=1}^{n}\sum_{j=1}^{q}(\bar{y}_{ij.} - \bar{y}_{.j.})^2$,

$SS_{Time} = nq\sum_{k=1}^{p}(\bar{y}_{..k} - \bar{y}_{...})^2$,

$SS_E = \sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk} - \bar{y}_{ij.} - \bar{y}_{..k} + \bar{y}_{...})^2$,

And

$\bar{y}_{...} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p} y_{ijk}}{nqp}$ is the overal mean .

$\bar{y}_{.j.} = \frac{\sum_{i=1}^{n}\sum_{k=1}^{p} y_{ijk}}{np}$ is the mean for group j

$\bar{y}_{ij.} = \frac{\sum_{k=1}^{p} y_{ijk}}{p}$ is the mean for the i^{th} subject in group j.

$\bar{y}_{..k} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q} y_{ijk}}{nq}$ is the mean for time k .

3. Estimation parameters

To estimate the parameters of the model we will use maximum likelihood method , which aims to make the likelihood function in the maximum. the likelihood function for the model (2-1) is:-

$$L(y;\mu,\tau_j,\Upsilon_k,\sigma_\delta^2,\sigma_e^2) = \pi_{i=1}^{n}\pi_{j=1}^{q}\pi_{k=1}^{p}\frac{1}{\sqrt{2\pi(\sigma_\delta^2+\sigma_e^2)}}\exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu-\tau_j-\Upsilon_k)^2}{2(\sigma_\delta^2+\sigma_e^2)}\right]$$

That is

$$L(y;\mu,\tau_j,\Upsilon_k,\sigma_\delta^2,\sigma_e^2) = [2\pi(\sigma_\delta^2+\sigma_e^2)]^{-\frac{nqp}{2}}\exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu-\tau_j-\Upsilon_k)^2}{2(\sigma_\delta^2+\sigma_e^2)}\right]$$

We can take the logarithm of both sides

$$\ln L(y;\mu,\tau_j,\Upsilon_k,\sigma_\delta^2,\sigma_e^2) = \left\{\ln[2\pi(\sigma_\delta^2+\sigma_e^2)]^{-\frac{nqp}{2}}\exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu-\tau_j-\Upsilon_k)^2}{2(\sigma_\delta^2+\sigma_e^2)}\right]\right\}$$

$$= -\frac{nqp}{2}\ln(2\pi) - \frac{nqp}{2}\ln(\sigma_\delta^2+\sigma_e^2) - \left[\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu-\tau_j-\Upsilon_k)^2}{2(\sigma_\delta^2+\sigma_e^2)}\right] \quad (3.1)$$

Now derive (3.1) for μ and equality the derivative to zero get:

$$\frac{\partial \ln L(y;\mu,\tau_j,\Upsilon_k,\sigma_\delta^2,\sigma_e^2)}{\partial\mu} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu-\tau_j-\Upsilon_k)}{(\sigma_\delta^2+\sigma_e^2)}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\hat{\mu}-\hat{\tau}_j-\hat{\Upsilon}_k) = 0$$

$$\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p} y_{ijk} - nqp\hat{\mu} = 0$$

$$\rightarrow \hat{\mu} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p} y_{ijk}}{nqp}$$

Volume : 6 | Issue : 9 | September 2016 | ISSN - 2249-555X | IF : 3.919 | IC Value : 74.50

$\therefore \quad \hat{\mu} = \bar{y}_{...}$,

we differentiate the (3.1) with respect to Υ_k and equate to zero:

$$\frac{\partial \ln L(y;\mu,\tau_j,\Upsilon_k,\sigma_\delta^2,\sigma_e^2)}{\partial \tau_j} = \frac{\sum_{i=1}^{n}\sum_{k=1}^{p}(y_{ijk} - \hat{\mu} - \hat{\tau}_j - \hat{\Upsilon}_k)}{(\sigma_\delta^2+\sigma_e^2)} = 0$$

$$\sum_{i=1}^{n}\sum_{k=1}^{p}(y_{ijk} - \hat{\mu} - \hat{\tau}_j - \hat{\Upsilon}_k) = 0$$

$$\sum_{i=1}^{n}\sum_{k=1}^{p}(y_{ijk} - np\hat{\mu} - np\hat{\tau}_j) = 0$$

$$\frac{y_{.j.}}{np} - \hat{\mu} - \hat{\tau}_j = 0$$

$$\therefore \hat{\tau}_j = \bar{y}_{.j.} - \bar{y}_{...} ,$$

we differentiate the (3.1) with respect to τ_j and equate to zero:

$$\frac{\partial \ln L(y;\mu,\tau_j,\Upsilon_k,\sigma_\delta^2,\sigma_e^2)}{\partial \Upsilon_k} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}(y_{ijk} - \hat{\mu} - \hat{\tau}_j - \hat{\Upsilon}_k)}{(\sigma_\delta^2+\sigma_e^2)} = 0$$

$$\sum_{i=1}^{n}\sum_{j=1}^{q}(y_{ijk} - \hat{\mu} - \hat{\tau}_j - \hat{\Upsilon}_k) = 0$$

$$\sum_{i=1}^{n}\sum_{j=1}^{q}(y_{ijk} - nq\hat{\mu} - nq\hat{\Upsilon}_k) = 0$$

$$\frac{y_{..k}}{np}\hat{\mu} - \hat{\Upsilon}_k = 0$$

$$\therefore \hat{\Upsilon}_k = \bar{y}_{..k} - \bar{y}_{...} ,$$

$$\sigma_e^2 = \frac{SS_E}{(p-1)(nq-1)}$$

$$\widehat{\sigma_e^2} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk} - \bar{y}_{ij.} - \bar{y}_{..k} + \bar{y}_{...})^2}{(p-1)(nq-1)}$$

we differentiate the (3.1) with respect to σ_δ^2 and equate to zero:

$$\frac{\partial \ln L(y;\mu,\tau_j,\Upsilon_k,\sigma_\delta^2,\sigma_e^2)}{\partial \sigma_\delta^2}$$

$$= \frac{-nqp}{2(\sigma_\delta^2+\sigma_e^2)} + \frac{2\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}\left(y_{ijk} - \mu - \tau_j - \Upsilon_k\right)^2}{4(\sigma_\delta^2+\sigma_e^2)^2}$$

$$\frac{-nqp(\sigma_\delta^2+\sigma_e^2) + \sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}\left(y_{ijk} - \hat{\mu} - \hat{\tau}_j - \hat{\Upsilon}_k\right)}{2(\sigma_\delta^2+\sigma_e^2)^2} = 0$$

$$-nqp(\sigma_\delta^2+\sigma_e^2) + \sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}\left(y_{ijk} - \hat{\mu} - \hat{\tau}_j - \hat{\Upsilon}_k\right) = 0$$

$$\widehat{\sigma_\delta^2} = \frac{\sum_{i=1}^{n}\sum_{k=1}^{p}(y_{ijk} - \hat{\mu} - \hat{\tau}_j - \hat{\Upsilon}_k)}{nqp} - \widehat{\sigma_e^2}$$

$$\widehat{\sigma_\delta^2} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}\left(y_{ijk} - \hat{\mu} - \hat{\tau}_j - \hat{\Upsilon}_k\right)}{nqp} - \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk} - \bar{y}_{ij.} - \bar{y}_{..k} + \bar{y}_{...})^2}{(p-1)(nq-1)}$$

Therefore

Volume : 6 | Issue : 9 | September 2016 | ISSN - 2249-555X | IF : 3.919 | IC Value : 74.50

$$\widehat{\sigma_\delta^2} + \widehat{\sigma_e^2} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\ \sum_{k=1}^{p}\ (y_{ijk} - \hat{\mu} - \hat{\tau}_j - \ \widehat{\Upsilon}_k)}{nqp}$$

4 . Some properties of Estimators in our model

We state some properties of estimators in repeated measurements model:

Unbiasedeness

Theorm (4.1) : The estimaters $\hat{\mu}$, $\hat{\tau}_j$ and $\widehat{\Upsilon}_k$ are unbiased for μ , τ_j , Υ_k respectively.

Proof :

Since $\hat{\mu} = \bar{y}_{...}$, $\rightarrow$ $E(\hat{\mu}) = E(\bar{y}_{...})$

$$E(\bar{y}_{...}) = E\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\ \sum_{k=1}^{p} y_{ijk}}{nqp}\right) \quad \text{then}$$

$$E(\hat{\mu}) = \frac{1}{nqp}\ E\left(\sum_{i=1}^{n}\sum_{j=1}^{q}\ \sum_{k=1}^{p} y_{ijk}\right)$$

$$= \frac{1}{nqp}\ E\left[\sum_{i=1}^{n}\sum_{j=1}^{q}\ \sum_{k=1}^{p}(\mu + \tau_j + \delta_{i(j)} + \ \Upsilon_k + e_{ijk})\right]$$

$$= \frac{1}{nqp}(nqp\mu) = \mu$$

$\therefore$ $E(\hat{\mu}) = \mu$

Now $\hat{\tau}_j = \bar{y}_{.j.} - \bar{y}_{...}$ $\rightarrow$ $E(\hat{\tau}_j) = E(\bar{y}_{.j.} - \bar{y}_{...})$

$E(\hat{\tau}_j) = E(\bar{y}_{.j.}) - E(\bar{y}_{...})$ since $E(\bar{y}_{...}) = \mu$

$$\therefore\ E(\hat{\tau}_j) = E\left(\frac{\sum_{i=1}^{n}\sum_{k=1}^{p} y_{ijk}}{np}\right) - \mu$$

$$= \frac{1}{np} E\left(\sum_{i=1}^{n}\sum_{k=1}^{p} y_{ijk}\right) - \mu$$

$$= \frac{1}{np} E\left[\sum_{i=1}^{n}\sum_{k=1}^{p}(\mu + \tau_j + \delta_{i(j)} + \ \Upsilon_k + e_{ijk})\right] - \mu$$

$$= \frac{1}{np}(np\mu + np\tau_j) - \mu$$

$$= \mu + \tau_j - \mu = \tau_j$$

$\therefore$ $E(\hat{\tau}_j = \tau_j$

Now $\widehat{\Upsilon}_k = \bar{y}_{..k} - \bar{y}_{...}$ $\rightarrow E(\widehat{\Upsilon}_k) = E(\bar{y}_{..k} - \bar{y}_{...})$

$$= E(\bar{y}_{..k}) - E(\bar{y}_{...})$$

$$= E\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{q} y_{ijk}}{nq}\right) - \mu$$

$$= \frac{1}{nq} E\left(\sum_{i=1}^{n}\sum_{j=1}^{q} y_{ijk}\right) - \mu$$

$$= \frac{1}{nq} E\left[\sum_{i=1}^{n}\sum_{j=1}^{q}(\mu + \tau_j + \delta_{i(j)} + \ \Upsilon_k + e_{ijk})\right] - \mu$$

$$= \frac{1}{nq}(nq\mu + nq\Upsilon_k) - \mu$$

$$= \mu + \Upsilon_k - \mu = \Upsilon_k$$

$\therefore E(\widehat{\Upsilon}_k) = \Upsilon_k$ □

Volume : 6 | Issue : 9 | September 2016 | ISSN - 2249-555X | IF : 3.919 | IC Value : 74.50

Consistency :

Theorem(4.2) : The estimators $\hat{\mu}$, $\hat{\tau}_j$, $\widehat{\Upsilon}_k$ are consistent of μ , τ_j , Υ_k respectively.

Proof: From theorem1 the Estimaters $\hat{\mu}$, $\hat{\tau}_j$ and $\widehat{\Upsilon}_k$ are unbiased .

Since $\hat{\mu} = \bar{y}_{...} \rightarrow var(\hat{\mu}) = var(\bar{y}_{...})$

$$var(\hat{\mu}) = var\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p} y_{ijk}}{nqp}\right)$$

$$= \frac{1}{n^2q^2p^2} var\left(\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p} y_{ijk}\right)$$

$$= \frac{1}{n^2q^2p^2}(nqp)(\sigma_\delta^2 + \sigma_e^2)$$

$$= \frac{\sigma_\delta^2+\sigma_e^2}{nqp} = \frac{\sigma_\delta^2+\sigma_e^2}{m} \rightarrow 0 \text{ as } m \rightarrow \infty \text{ , where } nqp=m$$

Then $\hat{\mu}$ is consistent of μ .

Since $\hat{\tau}_j = \bar{y}_{.j.} - \bar{y}_{...} \rightarrow var(\hat{\tau}_j) = var(\bar{y}_{.j.} - \bar{y}_{...})$

$$var(\hat{\tau}_j) = var(\bar{y}_{.j.}) + var(\bar{y}_{...})$$

$$= var\left(\frac{\sum_{i=1}^{n}\sum_{k=1}^{p} y_{ijk}}{np}\right) + \frac{\sigma_\delta^2+\sigma_e^2}{nqp}$$

$$= \frac{1}{n^2p^2}(np)(\sigma_\delta^2 + \sigma_e^2) + \frac{\sigma_\delta^2+\sigma_e^2}{nqp}$$

$$= \frac{(\sigma_\delta^2+\sigma_e^2)}{np} + \frac{\sigma_\delta^2+\sigma_e^2}{nqp}$$

$$= \frac{(q+1)(\sigma_\delta^2+\sigma_e^2)}{nqp} = \frac{(q+1)(\sigma_\delta^2+\sigma_e^2)}{m} \rightarrow 0 \text{ as } m \rightarrow \infty$$

Then $\hat{\tau}_j$ is consistent of τ_j .

Since $\widehat{\Upsilon}_k = \bar{y}_{..k} - \bar{y}_{...} \rightarrow var(\widehat{\Upsilon}_k) = var(\bar{y}_{..k} - \bar{y}_{...})$

$$var(\widehat{\Upsilon}_k) = var(\bar{y}_{..k}) + var(\bar{y}_{...})$$

$$= var\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{q} y_{ijk}}{nq}\right) + \frac{\sigma_\delta^2+\sigma_e^2}{nqp}$$

$$= \frac{1}{n^2q^2} var\left(\sum_{i=1}^{n}\sum_{j=1}^{q} y_{ijk}\right) + \frac{\sigma_\delta^2+\sigma_e^2}{nqp}$$

$$= \frac{1}{n^2q^2}(nq)(\sigma_\delta^2 + \sigma_e^2) + \frac{\sigma_\delta^2+\sigma_e^2}{nqp}$$

$$= \frac{(\sigma_\delta^2+\sigma_e^2)}{nq} + \frac{\sigma_\delta^2+\sigma_e^2}{nqp}$$

$$= \frac{(p+1)\sigma_\delta^2+\sigma_e^2}{nqp} = \frac{(p+1)\sigma_\delta^2+\sigma_e^2}{m} \rightarrow 0 \text{ as } m \rightarrow \infty$$

Then $\widehat{\Upsilon}_k$ is consistent of Υ_k . □

Efficiency

Volume : 6 | Issue : 9 | September 2016 | ISSN - 2249-555X | IF : 3.919 | IC Value : 74.50

Theorem(4.3) : The estimators
$\hat{\mu}$, $\hat{\tau}_j$, $\hat{\Upsilon}_k$ are efficietent of μ, τ, Υ, respectively.

Proof:

To prove that $\hat{\mu}$ is efficient we must show that $var(\hat{\mu})$ coincides with Cramer-Rao lower bound(C.R.L.B)

$$f(y;\mu,\tau_j,\Upsilon_k,\sigma_\delta^2,\sigma_e^2) = \frac{1}{\sqrt{2\pi(\sigma_\delta^2+\sigma_e^2)}} \exp\left[\frac{-(y_{ijk}-\mu-\tau_j-\Upsilon_k)^2}{2(\sigma_\delta^2+\sigma_e^2)}\right]$$

$$f(y;\mu,\tau_j,\Upsilon_k,\sigma_\delta^2,\sigma_e^2) = [2\pi(\sigma_\delta^2+\sigma_e^2)]^{-\frac{1}{2}} \exp\left[\frac{-(y_{ijk}-\mu-\tau_j-\Upsilon_k)^2}{2(\sigma_\delta^2+\sigma_e^2)}\right]$$

$$\therefore \ln f(y;\mu,\tau_j,\Upsilon_k,\sigma_\delta^2,\sigma_e^2) = -\frac{1}{2}\ln[2\pi(\sigma_\delta^2+\sigma_e^2)] - \frac{(y_{ijk}-\mu-\tau_j-\Upsilon_k)^2}{2(\sigma_\delta^2+\sigma_e^2)}$$

$$\frac{\partial \ln f}{\partial \mu} = \frac{y_{ijk}-\mu-\tau_j-\Upsilon_k}{\sigma_\delta^2+\sigma_e^2} \quad \to \quad \frac{\partial^2 \ln f}{\partial \mu^2} = -\frac{1}{\sigma_\delta^2+\sigma_e^2}$$

$$C.R.L.B = \frac{\acute{\tau}(\mu)}{-nqpE\left[\frac{\partial^2 \ln f(y;\mu,\tau_j,\Upsilon_k,\sigma_\delta^2,\sigma_e^2)}{\partial \mu^2}\right]} \quad , \quad \because \tau(\mu) = \mu \to \acute{\tau}(\mu) = 1$$

$$C.R.L.B = \frac{1}{-nqpE\left[-\frac{1}{\sigma_\delta^2+\sigma_e^2}\right]} = \frac{1}{\frac{nqp}{\sigma_\delta^2+\sigma_e^2}} = \frac{\sigma_\delta^2+\sigma_e^2}{nqp}$$

since $var(\hat{\mu}) = \frac{\sigma_\delta^2+\sigma_e^2}{nqp}$

$\therefore$ Thus $\hat{\mu}$ is an efficient estimator.

To prove that $\hat{\tau}_j$is efficient estimatr we must show that $var(\hat{\tau}_j)$ coincides with Cramer-Rao lower bound

$$\frac{\partial \ln f}{\partial \tau_j} = \frac{y_{ijk}-\mu-\tau_j-\Upsilon_k}{\sigma_\delta^2+\sigma_e^2}$$

$$\frac{\partial^2 \ln f}{\partial {\tau_j}^2} = -\frac{1}{\sigma_\delta^2+\sigma_e^2}$$

$$C.R.L.B = \frac{\acute{\tau}(\tau_j)}{-\frac{nqp}{(q+1)}E\left[\frac{\partial^2 \ln f(y;\mu,\tau_j,\Upsilon_k,\sigma_\delta^2,\sigma_e^2)}{\partial {\tau_j}^2}\right]} \quad , \quad \because \tau(\tau_j) = \tau_j \to \acute{\tau}(\tau_j) = 1$$

$$C.R.L.B = \frac{1}{-\frac{nqp}{(q+1)}E\left[-\frac{1}{(\sigma_\delta^2+\sigma_e^2)}\right]} = \frac{1}{\frac{nqp}{(q+1)\sigma_\delta^2+\sigma_e^2}} = \frac{(q+1)(\sigma_\delta^2+\sigma_e^2)}{nqp}$$

Since $var(\hat{\tau}_j) = \frac{(q+1)(\sigma_\delta^2+\sigma_e^2)}{nqp}$

Thus $\hat{\tau}_j$is an efficient estimator . □

To prove that $\hat{\Upsilon}_k$ is efficient estimatr we must show that $var(\hat{\Upsilon}_k)$ coincides with Cramer-Rao lower bound

$$\frac{\partial \ln f}{\partial \Upsilon_k} = \frac{y_{ijk}-\mu-\tau_j-\Upsilon_k}{\sigma_\delta^2+\sigma_e^2}$$

Volume : 6 | Issue : 9 | September 2016 | ISSN - 2249-555X | IF : 3.919 | IC Value : 74.50

$$\frac{\partial^2 \ln f}{\partial \Upsilon_k{}^2} = -\frac{1}{\sigma_\delta^2+\sigma_e^2}$$

$$\text{C.R.L.B}=\frac{\acute{\tau}(\Upsilon_k)}{-\frac{nqp}{(p+1)}E\left[\frac{\partial^2 \ln f\left(y;\mu,\tau_j,\Upsilon_k,\sigma_\delta^2,\sigma_e^2\right)}{\partial \Upsilon_k{}^2}\right]} \quad , \quad \because\ \tau(\Upsilon_k) = \Upsilon_k \ \rightarrow \ \acute{\tau}(\Upsilon_k) = 1$$

$$\text{C.R.L.B}=\frac{1}{-\frac{nqp}{(p+1)}E\left[-\frac{1}{(\sigma_\delta^2+\sigma_e^2)}\right]} = \frac{1}{\frac{nqp}{(p+1)\sigma_\delta^2+\sigma_e^2}} = \frac{(p+1)(\sigma_\delta^2+\sigma_e^2)}{nqp}$$

Since $\text{var}(\widehat{\Upsilon}_k) = \frac{(p+1)(\sigma_\delta^2+\sigma_e^2)}{nqp}$

Thus $\widehat{\Upsilon}_k$ is an efficient estimator . □

The likelihood-ratio test

Consider $H_\circ : \mu = 0$ versus $H_1 : \mu \neq 0$

The likelihood functions, denoted by $L(\omega)$ and $L(\Omega)$ are respectively

$$L(\omega) = \left[2\pi\left(\sigma_\delta^2+\sigma_e^2\right)\right]^{-\frac{nqp}{2}} \exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\tau_j-\Upsilon_k)^2}{2\left(\sigma_\delta^2+\sigma_e^2\right)}\right]$$

$$L(\Omega) = \left[2\pi\left(\sigma_\delta^2+\sigma_e^2\right)\right]^{-\frac{nqp}{2}} \exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu-\tau_j-\Upsilon_k)^2}{2\left(\sigma_\delta^2+\sigma_e^2\right)}\right]$$

Consider the problem of maximizing $L(\omega)$ and $L(\Omega)$ to a ccomplish this,
Where

$$\bar{y}_{...} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p} y_{ijk}}{nqp}, \bar{y}_{.j.} = \frac{\sum_{i=1}^{n}\sum_{k=1}^{p} y_{ijk}}{np}, \ \bar{y}_{..k} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q} y_{ijk}}{nq}$$

$$\hat{\mu} = \bar{y}_{...}, \hat{\tau}_j = \bar{y}_{.j.} - \bar{y}_{...}, \widehat{\Upsilon}_k = \bar{y}_{..k} - \bar{y}_{...},$$

$$\left(\widehat{\sigma_\delta^2} + \widehat{\sigma_e^2}\right) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk} - \bar{y}_{.j.} - \bar{y}_{..k} + \bar{y}_{...})^2}{nqp}$$

Let $\lambda = \frac{L(\widehat{\omega})}{L(\widehat{\Omega})}$

$$= \frac{\exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}+\bar{y}_{...}-\bar{y}_{..k}+\bar{y}_{...})^2}{2(\sigma_\delta^2+\sigma_e^2)}\right]}{\exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}{\frac{2\left(\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2\right)}{nqp}}\right]} =$$

$$\frac{\exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}+\bar{y}_{...}-\bar{y}_{..k}+\bar{y}_{...})^2+nqp\bar{y}_{...}{}^2}{\frac{2\left(\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2\right)}{nqp}}\right]}{\exp\left[\frac{-nqp}{2}\right]}$$

$$\lambda=\exp\left[\frac{-nqp}{2}\left(1+\frac{nqp\bar{y}_{...}{}^2}{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}\right)+\frac{nqp}{2}\right]$$

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$$\ln\lambda = \frac{-(nqp\bar{y}_{...})^2}{2\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}$$

$$-2\ln(\lambda) = \frac{(nqp\bar{y}_{...})^2}{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}$$

Since the function -2lnλ is a decreasing function, if follows that the critical region of the likelihood-ratio test can also be expressed in the form

$c_1 = \{x: -2\ln\lambda \geq c\}$, writinq $\Lambda(x) = -2\ln\lambda \longrightarrow c_1 = \{x: \Lambda(x) \geq c\}$

$$\bar{y}_{...} \sim N\left(\mu, \frac{\sigma_\delta^2+\sigma_e^2}{nqp}\right)$$

$$\frac{\bar{y}_{...}-0}{\sqrt{\sigma_\delta^2+\sigma_e^2}} \sim N(0,1)$$

$$\frac{(nqp\bar{y}_{...})^2}{\sigma_\delta^2+\sigma_e^2} \sim \chi^2_{(1)}$$

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}{\sigma_\delta^2+\sigma_e^2} \sim \chi^2_{(q-1)(p-1)}$$

The test of the composite hypothesis based on an F statistic with 1 and (q-1)(p-1) degrees of freedom.

Consider $H_\circ : \tau_j = 0$ versus $H_1 : \tau_j \neq 0$, j=1,.....,q

The likelihood functions, denoted by L(ω) and L(Ω) are respectively

$$L(\omega) = \left[2\pi(\sigma_\delta^2+\sigma_e^2)\right]^{-\frac{nqp}{2}} \exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu-\Upsilon_k)^2}{2(\sigma_\delta^2+\sigma_e^2)}\right]$$

$$L(\Omega) = \left[2\pi(\sigma_\delta^2+\sigma_e^2)\right]^{-\frac{nqp}{2}} \exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu-\tau_j-\Upsilon_k)^2}{2(\sigma_\delta^2+\sigma_e^2)}\right]$$

Let $\lambda = \frac{L(\widehat{\omega})}{L(\widehat{\Omega})}$

$$= \frac{\exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{..k})^2}{\frac{2\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}{nqp}}\right]}{\exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}{\frac{2\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}{nqp}}\right]}$$

$$= \frac{\exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2+\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(\bar{y}_{.j.}-\bar{y}_{...})^2}{\frac{2\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}{nqp}}\right]}{\exp\left[\frac{-nqp}{2}\right]}$$

$$\lambda = \exp\left[\frac{-nqp}{2}\left(1+\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(\bar{y}_{.j.}-\bar{y}_{...})^2}{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}\right)+\frac{nqp}{2}\right]$$

$$-2\ln\lambda = \frac{nqp\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(\bar{y}_{.j.}-\bar{y}_{...})^2}{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}$$

$$-2\ln\lambda = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}[\sqrt{nqp}(y_{.j.}-\bar{y}_{...})]^2}{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}$$

Since the function -2lnλ is a decreasing function, if follows that the critical region of the likelihood-ratio test can also be expressed in the form

$$c_1 = \{x: -2\ln\lambda \geq c\}.$$

Writinq $\wedge(x) = -2\ln\lambda \longrightarrow c_1 = \{x:\wedge(x) \geq c\}$

The test of the composite hypothesis based on an F statistic with (q-1) and (q-1)(p-1) degrees of freedom.

Consider $H_\circ : \gamma_k = 0$ versus $H_1 : \gamma_k \neq 0$, k=1,.....,p

The likelihood functions, denoted by L(ω) and L(Ω) are respectively

$$L(\omega) = [2\pi(\sigma_\delta^2+\sigma_e^2)]^{-\frac{nqp}{2}} \exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu-\tau_j)^2}{2(\sigma_\delta^2+\sigma_e^2)}\right]$$

$$L(\Omega) = [2\pi(\sigma_\delta^2+\sigma_e^2)]^{-\frac{nqp}{2}} \exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu-\tau_j-\gamma_k)^2}{2(\sigma_\delta^2+\sigma_e^2)}\right]$$

Let $\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})}$

$$= \frac{\exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{..k})^2}{\frac{2\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}{nqp}}\right]}{\exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}{\frac{2\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}{nqp}}\right]}$$

$$= \frac{\exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2+\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(\bar{y}_{..k}-\bar{y}_{...})^2}{\frac{2\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}{nqp}}\right]}{\exp\left[\frac{-nqp}{2}\right]}$$

$$\lambda = \exp\left[\frac{-nqp}{2}\left(1+\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(\bar{y}_{..k}-\bar{y}_{...})^2}{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}\right)+\frac{nqp}{2}\right]$$

$$-2\ln\lambda = \frac{nqp\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(\bar{y}_{..k}-\bar{y}_{...})^2}{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}$$

$$-2\ln\lambda = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(\sqrt{nqp}\bar{y}_{..k}-\bar{y}_{...})]^2}{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\bar{y}_{.j.}-\bar{y}_{..k}+\bar{y}_{...})^2}$$

Since the function -2lnλ is a decreasing function, if follows that the critical region of the likelihood-ratio test can also be expressed in the form

$$c_1 = \{x: -2\ln\lambda \geq c\}.$$

Writinq $\wedge(x) = -2\ln\lambda \longrightarrow c_1 = \{x: \wedge(x) \geq c\}$

The test of the composite hypothesis based on an F statistic with (p-1) and (q-1)(p-1) degrees of freedom.

5. Conclusion

The conclusions which are obtained throughout this work are given as follows :

1- The maximum likelihood estimetors for the parameters of one-way RMM are $\hat{\mu} = \bar{y}_{\ldots}$ $\hat{\tau}_j = \bar{y}_{.j.} - \bar{y}_{\ldots}$ $\hat{\Upsilon}_k = \bar{y}_{..k} - \bar{y}_{\ldots}$.

2-The estimators $\hat{\mu}$, $\hat{\tau}_j$, $\hat{\Upsilon}_k$ are unbiased,consistent and efficietent .

3-The likelihood- ratio test for the parameters μ,τ and Υ of one-way RMM is given.

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