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STABILITY OF THE FUNCTIONAL EQUATION

KEYWORDS functional equation, Hyres–Ulam stability, generalized Metric Space and fixed point approximation.

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ABSTRACT *Hilbert space and banach space are well known examples for Topological vector space.* In this paper, the authors] investigate the general solution of new cubic functional equation, $f(x+y+xy)=f(x)+f(y)+xf(y)+yf(x)$ and discuss its generalized Hyres – Ulam stability, via, fixed point approach.

1.1 INTRODUCTION

Consider the functional equation which defines multiplicative derivations in algebras,

$$f(x)=xf(y)+yf(x). \qquad (1.1)$$

It is easy to see that the real-valued function $f(x)=x$ in X is a solution of the functional equation (1.1) on the interval $(0,\infty)$.

During the 34^{th} International Symposium on Functional Equations, Gy. Maksa [1], posed the problem concerning the Hyers-Ulam Stability on the interval $(0,1]$ for the functional equation (1.1), and J.Tabor gave an answer to the equation of Maksa in [5]. On the other hand, Zs.Pales [2] remarked that the functional equation (1.1) on the interval $[1,\infty)$ for real-valued functions is superstable.

The following functional equation is obtained from the functional equation (1.1)

$$f(x+y+xy)=f(x)+f(y)+xf(y)+yf(x) \qquad (1.2)$$

In this chapter, we solve the functional equation (1.2) and then, by following the ideas of J.Tabor [5]and Zs.Pales [2], the Hyers-Ulam stability on the interval $(-1,0]$ and the superstability on the interval $[0,\infty)$ of the functional equation (1.2) will be investigated, respectively.

Throughout this chapter, we denote $\mathbf{R}$ and be the sets of real numbers and set of positive integers, respectively.

1.2 SOLUTIONS OF EQUATION (1.2)

It is easy to see that the real-valued function $f(x)=(x+1)\ln(x+1)$ is a solution of the functional equation (1.2) on the interval $(-1,\infty)$. In the following theorem, we will find out the general solution of the functional equation (1.2) on the interval $(-1,\infty)$.

Theorem 1.1

Let X be a real (or complex) linear space. A function $f:(-1,\infty)\to X$

Satisfies the functional equation (1.2) for all $x\in(-1,\infty)$ if and only if there exists a solution $D:(0,\infty)\to X$ of the functional equation (1.1),Such that

$$f(x)=D(x+1) \text{ for all } x\in(-1,\infty).$$

Proof.

Necessity. Let us define the mapping $D:(0,\infty)\to X$ by $D(x):=f(x-1)$. We claim that D is a solution of the functional equation (1.1). Indeed, for all $x,y\in(0,\infty)$, we have

$$D(xy)=f(xy-1)=f((x-1)+(y-1)+(x-1)(y-1))$$
$$=f(x-1)+f(y-1)+(x-1)f(y-1)+(y-1)f(x-1)$$
$$=xD(y)+yD(x).$$

Therefore D is a solution of the functional equation (1.1), as claimed, and $f(x)=D(x+1)$ for all $x\in(-1,\infty)$ .Sufficiency. This is obvious.

1.3 HYERS -ULAM STABILITY OF EQUATION (1.2)

We first state a theorem of F. Skof [3] concerning the stability of the additive functional equation $f(x+y)=f(x)+f(y)$ on a restricted domain.

Theorem 1.2

Let X be a real (or complex) Banach space. Given $c>0$, let a mapping $f:[0,c)\to X$ satisfy the inequality

$$\| f(x+y)-f(x)-f(y)\| \le \delta$$

for some $\delta \ge 0$ and for all $x,y\in[0,c)$ with $x+y\in[0,c)$. Then there exists an additive mapping $A:\mathbf{R}\to X$ such that

$$\| f(x)-A(x)\| \le 3\delta \text{ for all } x\in[0,c).$$

Our main theorem in this section is the Hyers-Ulam stability on the interval $(-1,0]$ of the functional equation (1.2) and the proof is similar to the one given in [5].

Theorem 1.3

Let X be a real (or complex) Banach space, and let $f:(-1,0]\to X$ be a mapping satisfying the inequality

$$\| f(x+y+xy)-f(x)-f(y)-xf(y)-yf(x)\| \le \delta \quad (1.3)$$

for some $\delta>0$ and for all $x,y\in(-1,0]$. Then there exists a solution $H:(-1,0]\to X$ of the functional equation (1.2) such that

$$\| f(x)-H(x)\| \le (4e)\delta \quad (1.4)$$

for all $x\in(-1,0]$.

Proof.

Let $g:(-1,0]\to X$ be a mapping defined by

$$g(x)=\frac{f(x)}{x+1}$$

for all $x\in(-1,0]$. Then, by (1.3), we see that g satisfy the inequality

$$\|g(x+y+xy)-g(y)-g(x)\| \le \frac{\delta}{(x+1)(y+1)}$$

for all $x,y\in(-1,0]$. If we define the mapping $F:[0,\infty)\to X$ by

$$F(-\ln(x+1))=g(x)$$

for all $x\in(-1,0]$, then by setting $u=-\ln(x+1)$ and $v=-\ln(y+1)$ we obtain

$$\| F(u+v)-F(u)-F(v)\| \le \delta e^{u+v} \quad (1.5)$$

for all $u,v\in[0.\infty)$. This means that

$$\| F(u+v)-F(u)-F(v)\| \le \delta e^{c}$$

for all $u,v\in[0,c)$ with $u+v<c$, where $c>1$ is an arbitrary given constant.

According to Theorem 4.2, there exists an additive mapping $A:\mathbf{R}\to X$ such that

$$\|F(u)-A(u)\| \le 3e\delta \quad (1.6)$$

for all $u\in[0,1]$. Moreover, it follows from (4.5) that

$$\| F(u+1)-F(u)-F(1)\| \le \delta e^{u+1}$$

$$\| F(u+2)-F(u+1)-F(1)\| \le \delta e^{u+2}$$

$$\vdots$$

$$\| F(u+k)-F(u+k-1)-F(1)\| \le \delta e^{u+k}$$

for all $u\in[0,1]$ and $k\in N$. Summing up these inequalities we obtain

$$\| F(u+k)-F(u)-kF(1)\| \le \delta e\cdot e^{u+k}\left(1+e^{-1}+.....+e^{-k+1}\right)$$

$$\le \delta e\cdot e^{u+k} \quad (1.7)$$

for all $u\in[0,1]$ and $k\in$. We assert that

$$\|F(v)-A(v)\| \le 4\delta e\cdot e^{v} \quad (1.8)$$

for all $v\in[0,\infty)$.

In fact, let $v\ge 0$ and let $k\in \quad \{0\}$ be given with $v-k\in[0,1]$. Then, by (1.6)and (1.7), we have

$$\| F(v)-A(v)\| \le \| F(v)-F(v-k)-kF(1)\|$$

$$+\|F(v-k)-A(v-k)\|+\|A(k)-kF(1)\|$$

$$\le \delta e\cdot e^{v}+3\delta e+k\|A(k)-kF(1)\|$$

$$\le \delta e\cdot e^{v}+3\delta e+k\|A(1)-F(1)\|$$

$$\le \delta e\cdot e^{v}+3\delta e+3\delta ev$$

$$\le \delta e(e^{v}+3(1+v))$$

$$\le 4\delta e\cdot e^{v}$$

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Hence, from (1.8) and the definition of F . It follows that

$$\|g(x)-A(-\ln(x+1))\| \le 4\delta e \cdot e^{-\ln(x+1)} = \frac{4\delta e}{x+1}$$

for all $x \in (-1,0]$, that is,

$$\left\|\frac{f(x)}{x+1} - A(-\ln(x+1))\right\| \le \frac{4\delta e}{x+1} \qquad (1.9)$$

for all $x \in (-1,0]$. If we put $H(x)=(x+1)A(-\ln(x+1))$ for all $x \in (-1,0]$, we can easily check that H is a solution of the functional equation (1.2) by Theorem 1.1. This and (1.9) yield that

$$\| f(x)-H(x)\| \le (4e)\delta$$

for all $x \in (-1,0]$ which proves (1.4). The proof of the theorem is complete.

1.4 SUPERSTABILITY OF EQUATION (1.2)

For the purpose, we will introduce the next result [4] which is essential to prove the main theorem.

Theorem 1.4

Let X be a real (or complex) Banach space, and let $c>0$ be a given constant. Suppose that a mapping $f:\mathbf{R} \to X$ satisfies teh inequality

$$\|f(x+y)-f(x)-f(y)\| \le \delta$$

for some $\delta \ge 0$ and for all $x,y \in \mathbf{R}$ with $|x|+|y|>c$. Then there exists a unique additive mapping $A:\mathbf{R} \to X$ such that

$$\|f(x)-A(x)\| \le 9\delta$$

for all $x \in \mathbf{R}$.

Now let us prove the main theorem of the section which is the superstability of the functional equation (1.2) on the interval $[0,\infty)$.

Therorem 1.5

Let X be a real (or complex) Banach space, and let $f:[0,\infty) \to X$ be a mapping satisfying the inequality,

$$\|f(x+y+xy)-f(x)-f(y)-xf(y)-yf(x)\| \le \delta \quad (1.10)$$

for some $\delta > 0$ and for all $x,y \in [0,\infty)$. Then f satisfies the functional equation (1.2) for all $x,y \in [0,\infty)$.

Proof.

Defining the mapping $g:[0,\infty) \to X$ by

$$g(x)=\frac{f(x)}{x+1}$$

for all $x \in [0,\infty)$ as in the proof of Theorem 1.3, and defining the mapping $F:[0,\infty) \to X$ by

$$F(\ln(x+1)) = g(x)$$

for all $x \in [0,\infty)$, we see by putting $u=\ln(x+1)$ and $v=\ln(y+1)$, that

$$\| F(u+v)-F(u)-F(v)\| \le \delta e^{-(u+v)} \qquad (1.11)$$

for all $u,v \in [0,\infty)$. We claim that F is additive.

From (1.1.1) with $\delta_n = \delta e^{-n}\,(n \in N)$, we obtain $\| F(u+v)-F(u)-F(v)\| \le \delta e$,

for all $u,v \in [0,\infty)$ with $u+v>n$.

We now define a mapping $T:\mathbf{R} \to X$ by

$$T(u)=\begin{cases} F(u)\ for\ u \ge 0, \\ -F(u)\ for\ u<0 \end{cases}$$

It is not difficult to see that

$$\|T(u+v)-T(u)-T(v)\| \le \delta_n$$

for all $u,v \in \mathbf{R}$ with $|u|+|v|>n$. Therefore, by Theorem 1.4 there exists a unique additive mapping $A_n:\mathbf{R} \to X$ satisfying

$$\|T(u)-A_n(u)\| \le 9\delta_n \qquad (1.12)$$

for all $u \in \mathbf{R}$. Let $m.n \in$ with $n>m$. Then the additive mapping $A_n:\mathbf{R} \to X$ satisfies $\|T(u)-A_n(u)\| \le 9\delta_n$ for all $u \in \mathbf{R}$. The uniqueness argument now implies $A_n = A_m$ for all $n \in$ with $n>m>0$, and thus $A_1 = A_2 = \ldots\ldots = A_n = \ldots\ldots$. Taking the limit in (1.12) as $n \to \infty$, we obtain $T = A_1$ and we deduce that F is additive.

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Now, according to the definitions of F and g, we have

$$\frac{f(x)}{x+1} = F\left(\ln(x+1)\right)$$

for all $x \in [0,\infty)$ that is,

$$f(x) = \left(x+1\right) F\left(\ln(x+1)\right)$$

for all $x \in [0,\infty)$, and hence we see that f satisfies the functional equation (1.2) for all $x, y \in [0,\infty)$ by Theorem 1.1 since F is additive and $D(x) = xF\left(\ln(x)\right)\left(x \in [1,\infty)\right)$ is a solution of the functional equation (1.1), This completes the proof of the theorem.

CONCLUSIONS

In this paper, the authors investigated the Generalized Hyers-Ulam-Rassias stability using direct method as well as using fixed point method. We find that the both the methods yields the same conclusion

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