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ON SOME NUMERICAL EXPERIMENTS WITH WEIERSTRASS ITERATIVE METHOD

KEYWORDS: Weierstrass method, polynomial zeros, error estimates, numerical experiments

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ABSTRACT One of the more effective methods for the simultaneous approximation of the roots of a polynomial is the Weierstrass method. In 2016, Proinov provide a detailed convergence analysis of the method and his results improve and generalize existing ones in the literature. To show some practical applications of Weierstrass iterative method, we develop code using computational system Wolfram Mathematica and Proinov's semilocal convergence theorem. Several numerical examples are provided. Furthermore, we give strong mathematical proof of experiment of Dochev and Byrnev (1964) and make a modification of their experiment with complex polynomial.

INTRODUCTION

Throughout this paper $(\mathbb{K},|\cdot|)$ denotes an algebraically closed normed field and $\mathbb{K}[z]$ denotes the ring of polynomials over $\mathbb{K}$. Let the vector space $\mathbb{K}^n$ be endowed with the p -norm $\|\cdot\|_p : \mathbb{K}^n \to \mathbb{R}_+$ defined by $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ for some $1 \le p \le \infty$.

In the literature there is a lot of iterative methods for simultaneous computation of all zeros of f. In 1891, Weierstrass [10] published his famous iterative method. The *Weierstrass method* is defined by the following iteration

$$x^{(k+1)} = x^{(k)} - W_f(x^{(k)}), \qquad k = 0,1,2,\ldots, \quad (1)$$

where the operator $W_f : D \subset \mathbb{K}^n \to \mathbb{K}^n$ is defined by

$$W_i(x) = \frac{f(x_i)}{a_0 \prod_{j \ne i} (x_i - x_j)} \qquad (i = 1,\ldots,n), \quad (2)$$

where $a_0 \in \mathbb{K}$ is the leading coefficient of f, the domain D of W_f is the set of all vectors in $\mathbb{K}^n$ with distinct components. The Weierstrass method (1) has second-order of convergence provided that all zeros of f are simple. Otherwise, it has only linear convergence.

In 1960–1966, the Weierstrass method (1) was rediscovered by Durand [4], Dochev [2], Kerner [5] and Prešić [8]. That is why it is known also as *'Dochev method'* or *'Weiersrass - Dochev method'* or *'Durand - Kerner method'* and so on. Since 1980, a number of authors have obtained semilocal convergence theorems for the Weierstrass method under different initial conditions. In 2016, Proinov [9], using a new approach for studding convergence of the iterative methods, present a theorem which generalizes and improves the all previous results in this area.

Throughout this paper we follow the terminology from Proinov [9]. For the sake of brevity, for a given p such that $1 \le p \le \infty$, we always denote by q the conjugate exponent of p, i.e. q is defined by means of $1 \le q \le \infty$ and $1/p + 1/q = 1$.

Proinov [9] study the semilocal convergence of the Weierstrass iteration with respect to the function of initial conditions $E_f : D \subset \mathbb{K}^n \to \mathbb{R}_+$ defined as follows

$$E_f(x) = \left\| \frac{W_f(x)}{d(x)} \right\|_p, \quad (3)$$

where the function $d : \mathbb{K}^n \to \mathbb{R}^n$ is defined by $d(x) = (d_1(x),\ldots,d_n(x))$, with $d_i(x) = \min_{j \ne i} |x_i - x_j|$ $(i = 1,\ldots,n)$.

Theorem 1 (Proinov [9]) Let $\mathbb{K}$ be a complete normed field, $f \in \mathbb{K}[z]$ be a polynomial of degree $n \ge 2$ and $1 \le p \le \infty$. Suppose $x^0 \in \mathbb{K}^n$ is an initial guess with distinct coordinates satisfying

$$E_f(x^0) < \frac{1}{2^{1/q}} \quad \text{and} \quad \phi(E_f(x^0)) \le 1, \quad (4)$$

where the function E_f is defined by (3) and the real function ϕ is defined by

$$\phi(t) = \frac{(n-1)^{1/q}\, t}{(1-t)(1-2^{1/q}t)} \left(1 + \frac{t}{(n-1)^{1/p}(1-2^{1/q}t)}\right)^{n-1}.$$

Then the following statements hold true.

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(i) CONVERGENCE. Starting from x^0, the Weierstrass iteration (1) is well-defined, remains in the closed ball $\bar{U}(x_0,\rho)$ and convergent to a root-vector ξ of f, where

$$\rho = \frac{\| W_f(x^0) \|}{1-\beta(E_f(x^0))}$$

and the real function β is defined by

$$\beta(t) = \frac{(n-1)^{1/q}t}{1-t}\left(1+\frac{t}{(n-1)^{1/p}(1-2^{1/q}t)}\right)^{n-1}.$$

Besides, the convergence is quadratic provided that $\phi(E_f(x^0))<1$.

(ii) A PRIORI ESTIMATE.. For all $k \ge 0$ we have the following estimate

$$\| x^k-\xi \| \neq \frac{\theta^k \lambda^{2^k-1}}{1-\theta\lambda^{2^k}} \| x^1 - x^0 \|,$$

where $\lambda = \phi(E_f(x^0))$ and $\theta = 1-2^{1/q}E_f(x^0)$.

(iii) FIRST A POSTERIORI ESTIMATE. For all $k \ge 0$ we have the following estimate

$$\| x^k-\xi \| \neq \frac{\| x^{k+1}-x^k \|}{1-\beta(E_f(x^k))}.$$

(iv) SeCOND A POSTERIORI ESTIMATE For all $k \ge 0$ we have the following estimate

$$\| x^{k+1}-\xi \| \neq \frac{\theta_k\lambda_k}{1-\theta_k\lambda_k^2} \| x^{k+1} - x^k \|,$$

where $\lambda_k = \phi(E_f(x^k))$ and $\theta_k = 1-2^{1/q}E_f(x^k)$.

(v) LOCALIZATION OF THE ZEROS. If $\phi(E_f(x^0))<1$ then the polynomial f has n simple zeros in $\mathbb{K}$. Moreover, for every $k \ge 0$ the closed disks

$$D_i^k = \{z \in \mathbb{K} : | z - x_i^{(k)} | \le r_i^k\} \quad (i=1,2,\ldots,n),$$

where $r_i^k = \frac{|W_i(x^k)|}{1-\beta(E_f(x^k))}$, are mutually disjoint and each of them contains exactly one zero of f.

The main advantages of this semilocal result are:

- weaker sufficient convergence conditions;
- computationally verifiable a posteriori error estimates;
- computationally verifiable sufficient conditions for all zeros of a polynomial to be simple;
- localization of the zeros.

In this paper we concentrate on procedures for the construction, analysis and practical application of Weierstrass iterative method with Proinov's theorem (2016) with the support of symbolic computation through several programs written in computer algebra system Wolfram Mathematica. We emphasize that the construction of presented method would be hardly feasible and most likely impossible without the use of this specific computer software. Two numerical examples are given to demonstrate convergence characteristics of the proposed method. Besides, we give strong mathematical proof of experiment of Dochev and Byrnev to integer polynomials (1964) and we present a modification of the experiment of Burnev and Dochev with complex polynomials.

ANALYSIS OF THE ALGO RITHM AND FLOW CHART

Now, we show the applicability of Theorem 1. Namely, if there exists an integer $m \ge 0$ such that

$$E_f(x^m) < \frac{1}{2^{1/q}} \quad \text{and} \quad \phi(E_f(x^m)) \le 1, \qquad (5)$$

then f has only simple zeros and the Weierstrass iteration (1) starting from x^0 is well defined and converges to a root-vector $\xi \in \mathbb{K}^n$ of f. Moreover, the method converges with order 2 to ξ provided that the second inequality in (5) is strict.

Besides, the following two a posteriori estimate errors hold:

$$\| x^k - \xi \| \le \max\{\varepsilon_k, \epsilon_k\}, \qquad (6)$$

where

$$\varepsilon_k = \frac{\| x^{k+1}-x^k \|_p}{1-\beta(E(x^k))} \quad \text{and} \quad \epsilon_k = \frac{\theta_k\lambda_k}{1-\theta_k\lambda_k^2} \| x^{k+1}-x^k \|_p.$$

In the examples below, we apply the Weierstrass iteration (1) using the stopping criterion

$$\max\{\varepsilon_k, \epsilon_k\} < 10^{-15} \qquad (k \ge m). \qquad (7)$$

For each example we calculate the smallest $m \ge 0$ which satisfies the convergence condition (5), the smallest $k \ge m$ for which the stopping criterion (7) is satisfied, as well as the value of ε_k and ϵ_k for the last k. From these data it follows that:

- f has only simple zeros;
- Weierstrass iteration (1) starting from x^0 is well-defined and converges with second-order to a root-vector of f;
- at kth iteration the zeros of f are calculated with an accuracy at least $\max\{\varepsilon_k, \epsilon_k\}$.

Today, mathematicians and computer scientists carry out sophisticated mathematical operations employing powerful computer machines supported by the modern computer algebra systems such as Mathematica, Maple, Axiom, GAP, Maxima,

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Sage and SymPy. These computer algebra systems, which enable both symbolic computation and a dynamic study using basins of attraction, are available on Windows, Mac OS X and Linux.

In this paper, to realize the proposed algorithm of Proinov, we develop several programs written in computer algebra system Mathematica.

In Figure 1 we presented the flow chart of the algorithm.

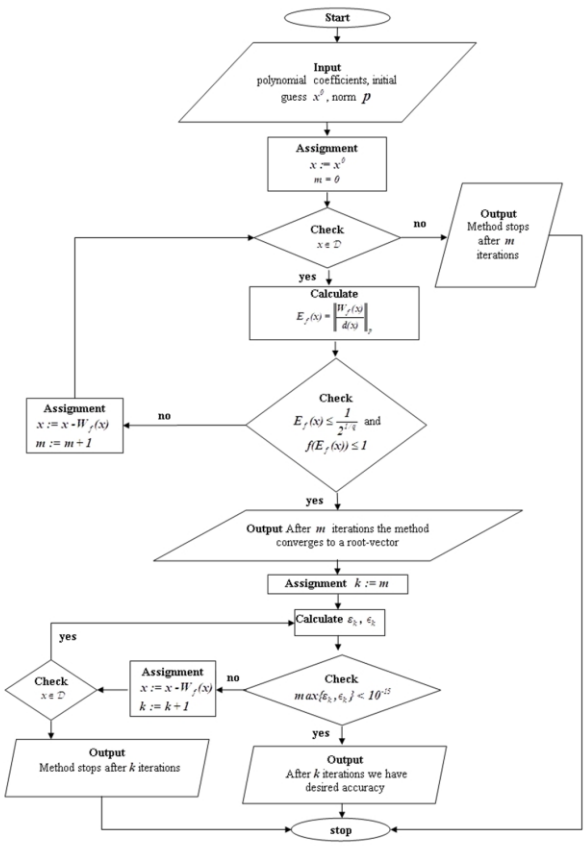

Figure 1: Flow chart of the algorithm

SOME NUMERICAL EXAMPLES

We provide some numerical examples to show some practical applications of the proposed method in the case $p = \infty$.

Example 1. We consider the polynomial

$$f(z) = z^5 - 15z^4 + 22z^3 + 438z^2 - 1175z - 1575$$

and the initial guess

$$x^0 = (-5.7, -1.8, 4.1, 6.2, 9.8)$$

which are taken from Nedzhibov et al. [6]. For this initial guess we have $E_f(x^0) = 0.408372$ and $\phi(E_f(x^0)) = 1636.760843$.

In this example we obtain that the convergence condition (5) is satisfied for $m = 2$ with the values for quantities $E_f(x^2) = 0.032277$ and $\phi(E_f(x^2)) = 0.163350$.

This guarantees that f has only simple zeros and the iteration (1) (starting from x^0) is well-defined and converges to a root-vector ξ of f with second order. Also, the stopping criterion (7) is satisfied for $k = 6$. Moreover, we can see that at the seventh iteration we have calculated the zeros of f with accuracy less than 10^{-66}.

In Table 1, we present the received error estimates for every one zero of the polynomial from fifth to seventh iteration.

TABLE – 1

Values of error estimates for Example 1

iter.	5	6	7
ε^1	7.234634×10^{-18}	3.621339×10^{-33}	2.655468×10^{-66}
ε^2	6.723266×10^{-18}	4.057853×10^{-33}	2.598090×10^{-68}
ε^3	3.621339×10^{-15}	3.841573×10^{-35}	1.768520×10^{-82}
ε^4	2.121631×10^{-20}	9.207271×10^{-48}	$2.404037 \times 10^{-108}$
ε^5	8.679440×102	5.222037×10^{-61}	$1.004064 \times 10^{-127}$
$\epsilon 1$	5.239813×10^{-32}	1.058180×10^{-65}	$7.051512 \times 10^{-132}$
$\epsilon 2$	4.869445×10^{-32}	1.646617×10^{-65}	$6.899147 \times 10^{-134}$
$\epsilon 3$	2.622819×10^{-29}	1.558854×10^{-67}	$4.696250 \times 10^{-148}$
$\epsilon 4$	1.536629×10^{-34}	3.736175×10^{-80}	$6.383846 \times 10^{-174}$
$\epsilon 5$	6.286240×10^{-42}	2.119026×10^{-93}	$2.666260 \times 10^{-193}$
$\max\{\varepsilon, \epsilon\}$	3.621339×10^{-15}	4.057853×10^{-33}	2.655468×10^{-66}

Example 2. In 2014, Petković et al. [7] consider the polynomial of the 21-st degree

$$f(z) = (z-4)(z^2-1)(z^4-16)(z^2+9)(z^2+16)$$
$$\times (z^2+2z+5)(z^2+2z+2)(z^2-2z+2)$$
$$\times (z^2-4z+5)(z^2-2z+10).$$

We consider this polynomial with Abert's initial approximation $x^0 \in \mathbb{C}^n$ (see [1]) given by

$$x_\nu^0 = -\frac{a_1}{n} + r_0 \exp(i\theta_\nu), \ \theta_\nu = \frac{\pi}{n}\left(2\nu - \frac{3}{2}\right), \ \nu = 1, \ldots, n, \tag{8}$$

where $n = 21$, $a_1 = -8$ and $r_0 = 5$. For this example we have $E_f(x^0) = 0.414509$ and $\phi(E_f(x^0)) = 4.069897 \times 10^{12}$. Here, we obtain that the convergence condition (5) is satisfied for $m = 20$ with the values for quantities $E_f(x^{20}) = 0.017438$ and $\phi(E_f(x^{20})) = 0.526174$. This guarantees that f has only simple zeros and the iteration (1) (starting from x^0) is well-defined and converges to a root-vector ξ of f with second order. Also, the stopping criterion (7) is satisfied for $k = 23$. Moreover, we can see that at the twenty fourth iteration we have calculated the zeros of f with accuracy less

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than 10^{-38} (see Table 2 with maximum values of error estimates).

TABLE – 2

Values of error estimates for Example 2

iteration	max ε	ϵ max
20	0.164430	0.112940
21	0.000394	3.132168×10^{-6}
22	1.593239×10^{-8}	5.076827×10^{-15}
23	7.195780×10^{-18}	1.035585×10^{-33}
24	8.277115×10^{-38}	1.370212×10^{-75}

In Figure 2, we present the trajectories of approximations generated by the method (1) after 26 iterations with $r_0 = 5$.

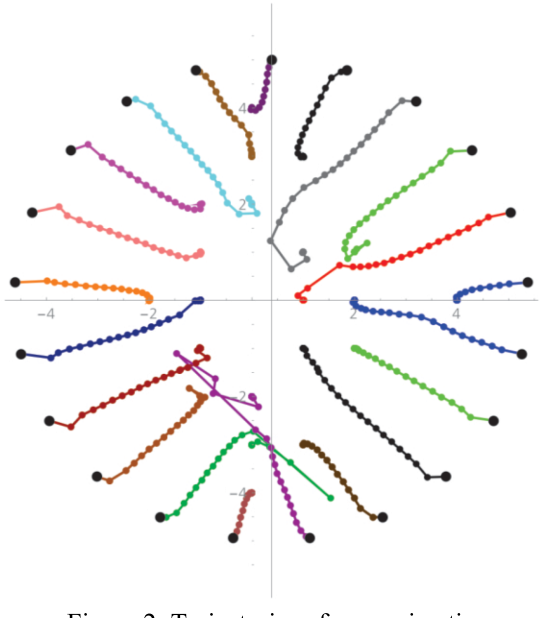

Figure 2: Trajectories of approximations

EXPERIMENT OF DOCHEV AND BYRNEV

In 1964 Dochev and Byrnev [3] proposed the following experiment:

Experiment .1 To investigate the convergence behavior of the Weierstrass method (1) for all polynomials of degree 4 with different integers roots in $[-10,10]$ and the initial guess $x^{(0)} = (-7.5, -2.5, 2.5, 7.5)$.

The number of the tested polynomials is 5985. After computerizing test they conclude that for two of the polynomials the initial guess is not suitable, since the Weierstrass operator is not defined after the first iteration. The polynomials are $f(z) = (z+10)(z+5)(z-5)(z-9)$ and $f(z) = (z+9)(z+5)(z-5)(z-10)$.

For every one of the others 5983 polynomials Dochev and Byrnev proved that the Weierstrass method converges. The approximation guess required to satisfy the stopping criterion in l_1 norm

$$\| x^{(k+1)} - x^{(k)} \| \le 10^{-3}, \qquad (9)$$

i.e. they used the condition (9) as criterion of convergence for the Weierstrass method. The average number of iterations required to satisfy the stopping criterion is $41801/5983 \approx 7$.

Unfortunately their proof cannot be accept as strong science proof of the experiment.

Our main purpose is to give strong science proof of the experiment of Dochev and Byrnev using Theorem 1 with our program, developing with computer system Wolfram Mathematica.

The result of our experiment is:

- The initial guess is not suitable for two of the polynomials (see Table 3);
- Theorem 1 guaranties the second order of convergence for the Weierstrass method for the rest 5983 polynomials.

The average number of iterations required to satisfy the initial condition (7) is $26665/5983 \approx 4$.

TABLE – 3

The Weierstrass method is not convergence

polynomial	initial guess
$f(z) = (z+10)(z+5) \times (z-5)(z-9)$	$x^{(1)} = (-9.21875, -8.96875, 8.59375, 8.59375)$
$f(z) = (z+9)(z+5) \times (z-5)(z-10)$	$x^{(1)} = (-8.59375, -8.59375, 8.96875, 9.21875)$

MODIFICATION OF EXPERIMENT OF DOCHEV AND BYRNEV

In this section we expand the experiment of Dochev and Byrnev, namely with investigation complex polynomial with randomly chosen initial guess.

Experiment 2 To investigate the convergence behavior of the Weierstrass method (1) for all polynomials of degree 4 with different complex roots $\alpha + \beta i$, where α and β are integer numbers such that $\alpha, \beta \in [-2,2]$ and randomly chosen initial guess x^0.

The number of the polynomials is 12650. All polynomials are tested with 1000 randomly chosen initial guesses from the square $S = \{-2, 2\} \times \{-2, 2\}$ centered at the origin.

The result of our experiment is that Theorem 1 guarantees the second order of convergence for the Weierstrass method for the every one of the investigated polynomials for every one of the 1000 different initial guess.

In the Table 4 we present the average values of the quantities $E_f(x)$, $\phi(E_f(x))$ and number of the iterations for the tested initial guess.

TABLE – 4
Average values of quantities for the modified Dochev and Byrnev experiment

initial guess	average $E_f(x)$	average $\phi(E_f(x))$	average number of iterations
1−100	0.060214244	0.321692975	7.938566008
101−200	0.059341226	0.313700698	7.246045566
201−300	0.059446438	0.314434758	7.713817578
301−400	0.059446417	0.314548326	7.577915065
401−500	0.059462812	0.314544624	7.629496443
501−600	0.059386827	0.313995092	7.241633202
601−700	0.059058719	0.312165984	7.830938345
701−800	0.059434172	0.314401586	7.452508559
801−900	0.059477533	0.314761018	7.401513834
901−1000	0.059427391	0.314378573	7.499177866

REFERENCES:

[1] Abert O., Iteration methods for finding all zeros of a polynomial simultaneously, Math. Comput. 27 (1973) 339–344.
[2] Dochev K., Modified Newton method for the simultaneous approximation of all roots of an algebraic equation, Phys. Math. J. Bulg. Acad. Sci. 5 (1962) 136–139. (in Bulgarian)
[3] Dochev K., P. Byrnev, Certain modifications of Newton's method for the approximate solution of algebraic equations, USSR Comput. Math. Math. Phys. 1964, 4(5), 174—182.
[4] Durand E., Solutions Numériques des Equations Algébriques, Tome I: Équations du Type F(x) = 0; Racines d'un Polynôme, Masson, Paris, 1960.
[5] Kerner I. O., Ein Gesamtschrittverfahren zur Berechnung der Nullstellen von Polynomen, Numer. Math. 8 (1966) 290–294.
[6] Nedzhibov G. H., M. G. Petkov, On a family of iterative methods for simultaneous extraction of all roots of algebraic polynomial, Appl. Math. Comput. 2005, 162, 427—433.
[7] Petković M. S., L. D. Petković, J. Dzunić, On an efficient simultaneous method for finding polynomial zeros, Appl. Math. Letters 28 (2014) 60–65.
[8] Prešić S. B., Un procédé itératif pour la factorisation des polynômes, C. R. Acad. Sci. Paris Sèr. A 262 (1966) A862–A863.
[9] Proinov P. D., General convergence theorems for iterative processes and applications to the Weierstrass root-finding method, J. Complexity 2016, 33, 118–144.
[10] Weierstrass K., Neuer Beweis des Satzes, dass jede ganze rationale Function einer Veränderlichen dargestellt werden kann als ein Product aus linearen Functionen derselben Veränderlichen, Sitzungsber, Königl. Akad. Wiss. Berlin, 1891; pp. 1085–1101.