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Mathematics

DYNAMICAL BEHAVIOR IN A FRACTIONAL ORDER EPIDEMIC MODEL

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ABSTRACT Fractional order SIR epidemic model is considered for dynamical analysis. The basic reproductive number is established and an analysis is carried out to study the stability of the equilibrium points. The time plots and phase portraits are provided for different sets of parameter values. Numerical simulations are presented to illustrate the stability analysis using Generalized Euler method.

KEYWORDS : Fractional order, SIR model, Differential equations, Stability, Generalized Euler Method.

INTRODUCTION

Fractional Calculus is a branch of Mathematics that deals with the study of integrals and derivatives of non-integer orders, plays an outstanding role and have found several applications in large areas of research during the last decade. Behavior of many dynamical systems can be described and studied using the fractional order systems. Fractional derivatives describe effects of memory. This section presents some important definitions of fractional calculus which arise as natural generalization of results from calculus [3].

Definition 1. The Riemann – Liouville fractional Integral of order $0 \le \alpha \le 1$ is defined as

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int (t-u)^{\alpha-1} f(u)\, du,\ t > 0 .$$

Definition 2. The Riemann – Liouville fractional Derivative is defined as $D_t^{\alpha} f(t) = \frac{d}{dt} J^{1-\alpha} f(t)$.

Definition 3. The Caputo fractional derivative is defined as $D_t^{\alpha} f(t) = J^{1-\alpha} \frac{d}{dt} f(t)$.

GENERALIZED TAYLORS FORMULA AND EULER METHOD

A generalization of Taylor's formula that involves Caputo fractional derivatives is presented in [7]. Suppose that $D^{k\alpha} f(x) \in C(0, a]$ for $k = 0, 1, \ldots, n+1$, where $0 < \alpha \le 1$. Then we have

$$f(x) = \sum_{i=0}^{n} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+) + \frac{(D^{(n+1)\alpha} f)(\zeta)}{\Gamma((n+1)\alpha+1)} x^{(n+1)\alpha}$$

where $0 \le \zeta \le x, \forall x \in C(0,a]$. For $\alpha = 1$, the generalized Taylor's formula reduces to the classical Taylor's formula.

In [6], Z.M. Odibat and Shaher Momani derived the generalized Euler's method for the numerical solution of initial value problems with caputo derivatives. The method is a generalization of the classical Euler's method. Consider the following general form of IVP; $D^{\alpha} y(t) = f(t, y(t)), y(0) = y_0$ for $0 < \alpha \le 1$, $0 < t < a$. The general formula for Generalized Euler's Method (GEM) is

$$y(t_{j+1}) = y(t_j) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} f(t_j, y(t_j)) \qquad (1)$$

where $j = 0, 1, \ldots, n-1$. It is clear that if $\alpha = 1$, then the generalized Euler's method (1) reduces to the classical Euler's method.

MODEL DESCRIPTION

Mathematical modeling is used to analyze, study the spread of infectious diseases and predict the outbreak and to formulate policies to control an epidemic. We obtain fractional SIR epidemic model by introducing fractional derivative of order $\alpha (0 < \alpha \le 1)$ in the classical SIR epidemic equations. In this paper, we study fractional order SIR epidemic model with vaccination and treatment. The total population **N** is partitioned into three compartments which are Susceptible, Infected and Recovered with sizes denoted by S(t), I(t) and R(t) [4].

Variable	Meaning
S(t)	Number of Susceptible Individuals at time t
I(t)	Number of Infectious Individuals at time t
R(t)	Number of Recovered Individuals at time t
Parameters	**Meaning**
b	Birth rate or Recruitment rate
d	Death rate
$\frac{1}{\delta}$	Infectious Period
β	Contact rate

Then the system leads to fractional equations given by

$$\begin{aligned} D^{\alpha}S(t) &= b-\beta S(t)I(t)-dS(t) \\ D^{\alpha}I(t) &= \beta S(t)I(t)-\delta I(t)-dI(t) \qquad (2) \\ D^{\alpha}R(t) &= \delta I(t)-dR(t) \end{aligned}$$

where the arbitrary order α in the sense of Caputo and $0<\alpha<1$. Also we have $D^{\alpha}N(t)=b-dN(t)$. Since $N(t)=S(t)+I(t)+R(t)$, $R(t)$ can always be obtained by the equation $R(t)=N(t)-S(t)-I(t)$.

We now consider the system of equations

$$\begin{aligned} D^{\alpha}S &= b-\beta SI-dS \\ D^{\alpha}I &= \beta SI-\delta I-dI \qquad (3) \\ D^{\alpha}N &= b-dN \end{aligned}$$

with the following initial conditions $S(0)=S_0,\ I(0)=I_0, N(0)=N_0$, where $0<\alpha<1$.

EQUILIBRIUM POINTS

Consider the System[9]

$$\begin{aligned} D_*^{\alpha}x_1(t) &= g_1(x_1,x_2,x_3) \\ D_*^{\alpha}x_2(t) &= g_2(x_1,x_2,x_3) \qquad (4) \\ D_*^{\alpha}x_3(t) &= g_3(x_1,x_2,x_3) \end{aligned}$$

with initial values $x_1(0)=x_{01},\ x_2(0)=x_{02},\ x_3(0)=x_{03}$.

To evaluate the equilibrium points, let $D_*^{\alpha}x_i(t)=0$. $g_i(x_1^*,x_2^*,x_3^*)=0$, $i=1,2,3$. from which we can get the equilibrium points x_1^*,x_2^*,x_3^*. The stability result for the fractional order linear system is given below.

Lemma 1. *[5] The fractional order autonomous system $D^{\alpha}x(t)=Ax(t), x(0)=x_0$ where $0<\alpha<1$, $x\in R^n$ and $A\in R^{n\times n}$ is*

(a) Locally asymptotically stable if and only if $|\arg(\lambda_i(A))|>\alpha\frac{\pi}{2},(i=1,2,\ \ ,n)$. Where $\arg(\lambda_i(A))$ denotes the argument of the eigenvalue λ_i of A.

(b) Stable if and only if $|\arg(\lambda_i(A))|\geq\alpha\frac{\pi}{2},(i=1,2,\ \ ,n)$.

Proposition 2. *The system of equations (3) has a disease free equilibrium point $\left(\frac{b}{d},0,\frac{b}{d}\right)$.*

Proof: To obtain the equilibrium points we consider $D^{\alpha}S=0,\ D^{\alpha}I=0,\ D^{\alpha}N=0$. In disease free situation, that is, there is no infection $I=0$. Therefore from the system of equations (3) we get $b-dS=0\Rightarrow s=\frac{b}{d}$.

Hence the Disease Free Equilibrium (DFE) Point is $(S,I,N)=\left(\frac{b}{d},0,\frac{b}{d}\right)$. Also the Endemic Equilibrium Point is $(S,I,N)=\left(\frac{\delta+d}{\beta},\frac{d}{\beta}(R_0-1),\frac{b}{d}\right)$.

BASIC REPRODUCTION NUMBER

Basic Reproduction Number R_0 is defined as "*The average number of secondary infectious caused by a single infectious individual during theirentire infectious lifetime*".

R_0 can be determined by Next Generation Matrix (NGM) approach[8]. The Next Generation Matrix is given by $K=FV^{-1}$ where,

$$F=\begin{bmatrix}\beta S & 0\\ 0 & 0\end{bmatrix}, V=\begin{bmatrix}\delta+d & -\beta I\\ \beta S & \beta I+d\end{bmatrix}$$

$$K=FV^{-1}=\begin{bmatrix}\dfrac{(\beta S_0)(\beta I_0+d)}{(\delta+d)(\beta I_0+d)+(\beta S_0)(\beta I_0)} & 0\\ 0 & 0\end{bmatrix} \qquad (5)$$

At Disease Free Equilibrium (DFE), $S_0 = \frac{b}{d}$ and $I_0 = 0$. Substituting in the Next Generation Matrix, we get $K = FV^{-1} = \begin{bmatrix} \frac{b\beta}{d(\delta+d)} & 0 \\ 0 & 0 \end{bmatrix}$. Since R_0 is the most dominant eigenvalue of Next Generation Matrix, we get $R_0 = \frac{b\beta}{d(\delta+d)}$.

Proposition 3. *The Disease Free Equilibrium is locally asymptotically stable if $0 < R_0 < 1$ and unstable if $R_0 > 1$.*

Proof: Based on the system of equations (3), the Jacobian Matrix is

$$J(S, I, N) = \begin{bmatrix} -\beta I - d & -\beta S & 0 \\ \beta I & \beta S - \delta - d & 0 \\ 0 & 0 & -d \end{bmatrix} \tag{6}$$

Suppose that $R_0 < 1$, the Jacobian matrix of the system of equations (3) at DFE $(S, I, N) = \left(\frac{b}{d}, 0, \frac{b}{d}\right)$

is given by $J(E_0) = \begin{bmatrix} -d & -\beta\frac{b}{d} & 0 \\ 0 & \beta\frac{b}{d} - \delta - d & 0 \\ 0 & 0 & -d \end{bmatrix}$

The eigenvalues are $\lambda_1 = \lambda_2 = -d$ and $\lambda_3 = \beta\frac{b}{d} - \delta - d$. Hence the Disease Free Equilibrium point of the system is locally asymptotically stable if $0 < R_0 < 1$ and $|\arg(\lambda_{1,2,3})| > \alpha\frac{\pi}{2}$.

Proposition 4. *The system of equations (3) has an endemic equilibrium point and asymptotically stable if $R_0 > 1$.*

Proof: From the system of equations (3), we have $S = \frac{\delta + d}{\beta}$, $I = \frac{d}{\beta}(R_0 - 1)$, $N = \frac{b}{d}$. The Jacobian matrix for the Endemic equilibrium point is

$$J(E_1) = \begin{bmatrix} -dR_0 & -\delta - d & 0 \\ d(R_0 - 1) & 0 & 0 \\ 0 & 0 & -d \end{bmatrix}.$$

The eigen values are $\lambda_1 = -d$ and $\lambda_{2,3} = \frac{-dR_0}{2} \pm \frac{1}{2}\sqrt{(dR_0)^2 - 4d(\delta + d)(R_0 - 1)}$. Hence the Endemic Equilibrium point of the system is locally asymptotically stable if $R_0 > 1$ and $|\arg(\lambda_{1,2,3})| > \alpha\frac{\pi}{2}$.

NUMERICAL EXAMPLES

Numerical solution of the fractional order system is

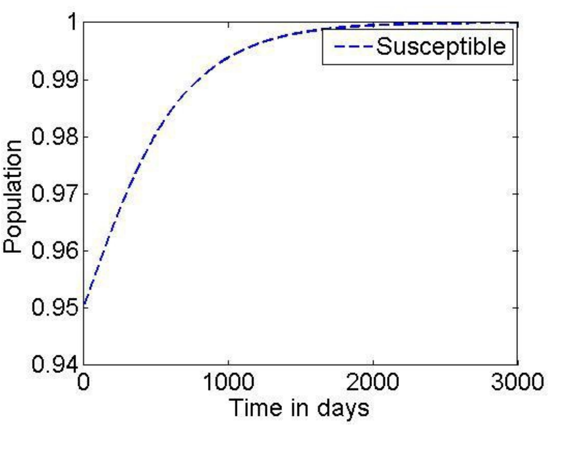
$$S(t_{j+1}) = S(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)}(b - \beta S(t_j) I(t_j) - dS(t_j))$$

$$I(t_{j+1}) = I(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)}(\beta S(t_j) I(t_j) - \delta I(t_j) - dI(t_j))$$

$$N(t_{j+1}) = N(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)}(b - dN(t_j))$$

for $j = 0, 1, 2, \ldots, k-1$. Numerical techniques are used to analyze the qualitative properties of fractional order differential equations since the equations do not have analytic solutions in general.

Example 1 *Let us consider the parameter values $b = 0.02$, $d = 0.02$, $\beta = 0.01$, $\delta = 0.02$ and $h = 0.1$* with *the initial conditions $S(0) = (0.95)$, $I(0) = (0.05)$ and $N(0) = (1.00)$, the fractional derivative order $\alpha = 0.9$. For these parameter the corresponding eigen values are $\lambda_{1,2} = -0.02$, $\lambda_3 = -0.03$ for E_0. Also $|\arg(\lambda_{1,2,3})| = 3.1416 > 1.4137 = \alpha\frac{\pi}{2}$ and $R_0 = 0.25 < 1$ then the disease free equilibrium is locally asymptotically stable. See Figure 1.*



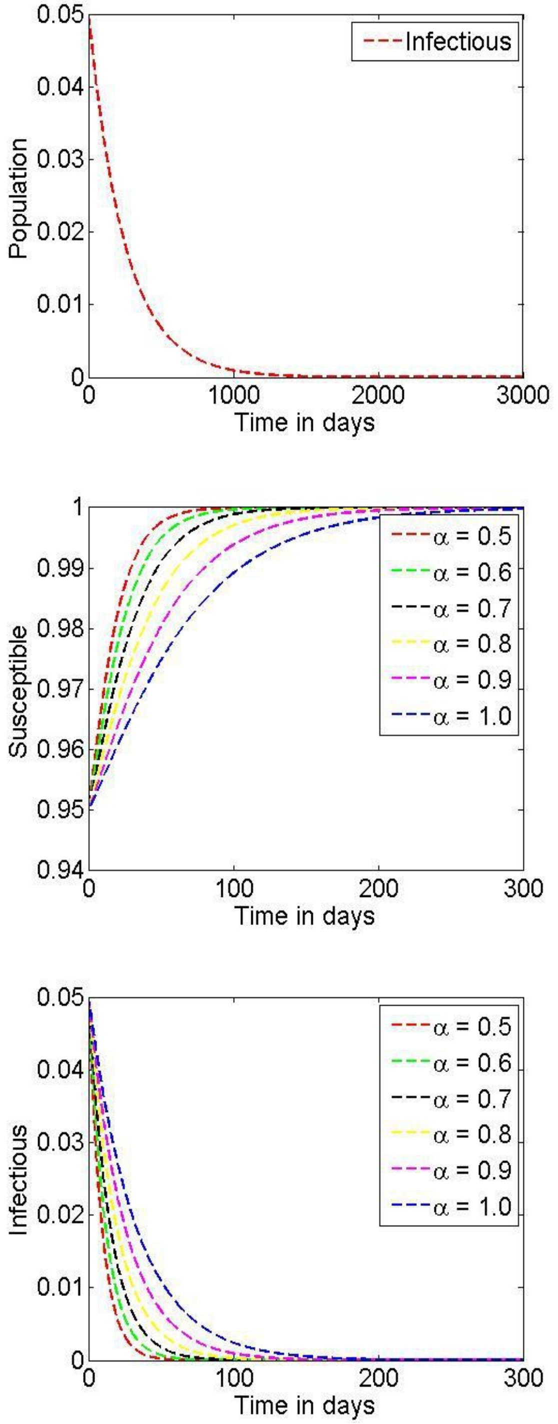
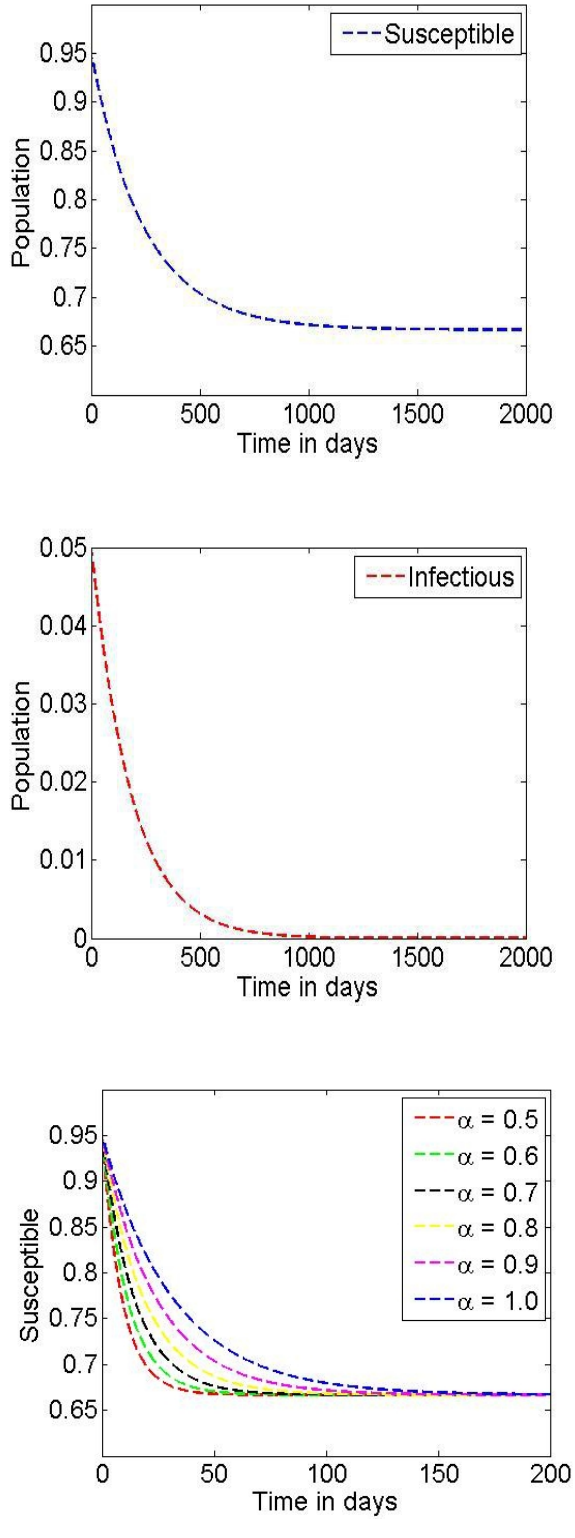


Figure 1. Time Series of disease free equilibrium E_0 and Different Fractional Derivatives $(\alpha's)$ $(R_0 < 1)$

Example 2. *Let us consider the parameter values* $b = 0.02,\ d = 0.03, \beta = 0.01, \delta = 0.02$, *and* $h = 0.1$, *with initial conditions* $(S(0) = (0.95),\ I(0) = (0.05)$ *and* $N(0) = (1.00)$, *the fractional derivative order* $\alpha = 0.9$. *For these parameter the eigenvalues are* $\lambda_{1,2} = -0.03, \lambda_3 = -0.0433$ *for* E_0. *Also* $\left|\arg(\lambda_{1,2,3})\right| = 3.1416 > 1.4137 = \alpha\ \frac{\pi}{2}$ *and* $R_0 = 0.1333 < 1$, *then the disease free equilibrium is locally asymptotically stable. See Figure 2.*



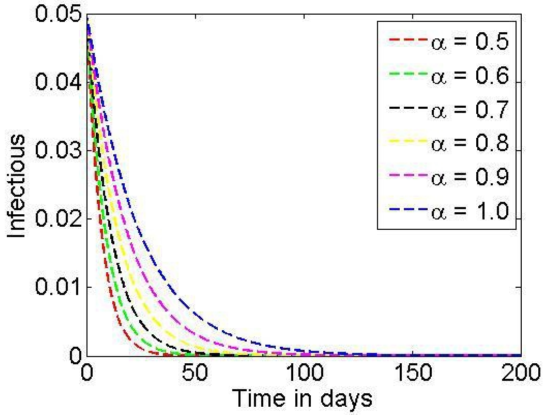
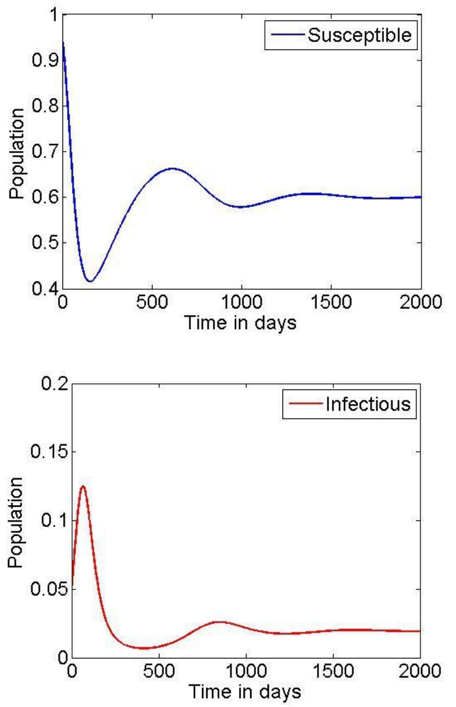


Figure 2. Time Series of disease free equilibrium E_0 and Different Fractional Derivatives ($\alpha's$) with $R_0 < 1$.

Example 3 *Let us consider the parameter values* $b = 0.025,\ d = 0.03, \beta = 0.6, \delta = 0.33$, *and* $h = 0.1$ *with initial conditions* $(S(0) = (0.95), I(0) = (0.05)$ *and* $N(0) = (1.00)$, *the fractional derivative order* $\alpha = 0.9$. *For these parameters the corresponding eigen values are* $\lambda_1 = -0.03$, *and* $\lambda_{2,3} = -0.0208 \pm 0.0614i$ *for* E_1.. *Also* $|\arg(\lambda_{1,2,3})| = 3.1416 > 1.4137 = \alpha\,\dfrac{\pi}{2}$ *and* $R_0 = 1.3889 > 1$. *Then the endemic equilibrium is locally asymptotically stable. See Figure3.*



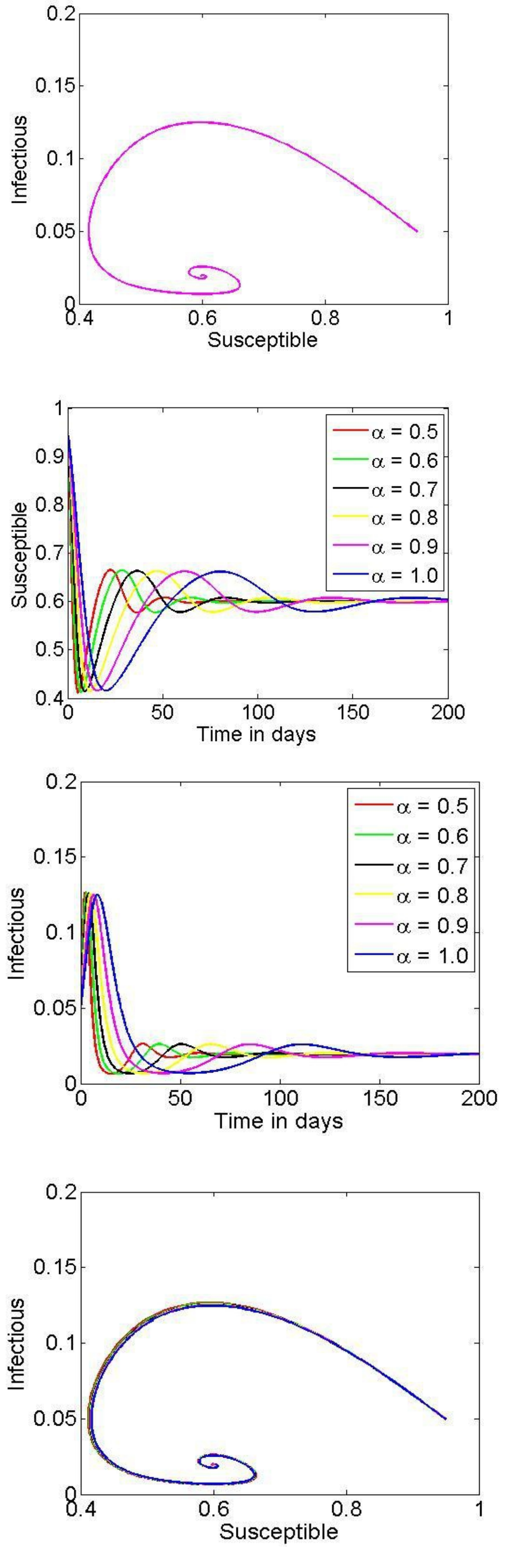


Figure 3. Time series and Phase diagram of endemic equilibrium E_1 and Different Fractional Derivatives ($\alpha's$) with $R_0 > 1$.

BIFURCATION

Bifurcation diagrams provide information about abrupt changes in the qualitative behavior in the dynamics of the system. The parameter values at which these changes occur are called bifurcation points. If the qualitative change occurs in a neighborhood of an equilibrium point or periodic solution, it is local bifurcation. In this section, we give the bifurcation diagrams of the systems (3). See Figure 4-7.

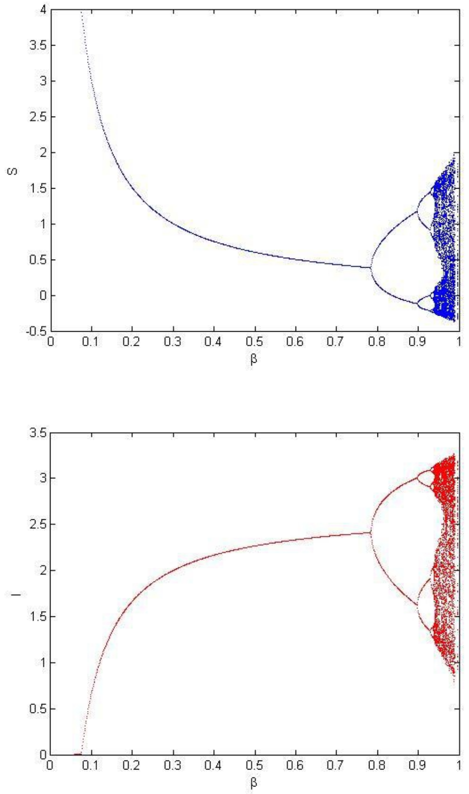
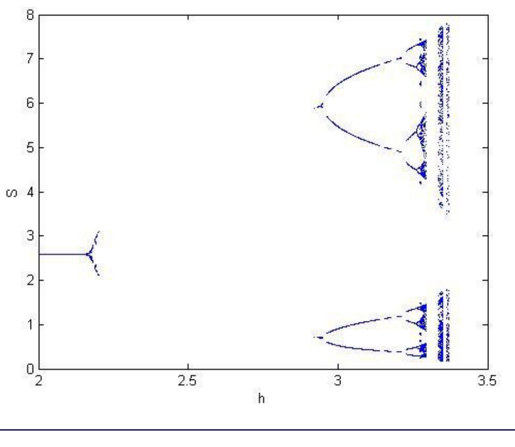


Figure 4. The bifurcation of Susceptible population, Infected population with initial values $(S_0, I_0) = (0.95, 0.05)$, $b = 0.8$, $d = 0.2$, $\delta = 0.1$, $h = 1.0$, $\beta \in [0.0, 1.0]$ and $\alpha = 0.5$.



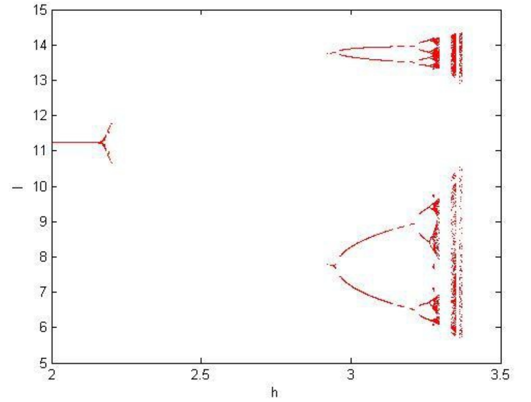


Figure 5. The bifurcation of Susceptible population, Infected population with initial values $(S_0, I_0) = (0.95, 0.05)$, $b = 4.0$, $d = 0.2$, $\beta = 0.12$, $\delta = 0.11$, $h \in [2.0, 3.5]$ and $\alpha = 0.5$.

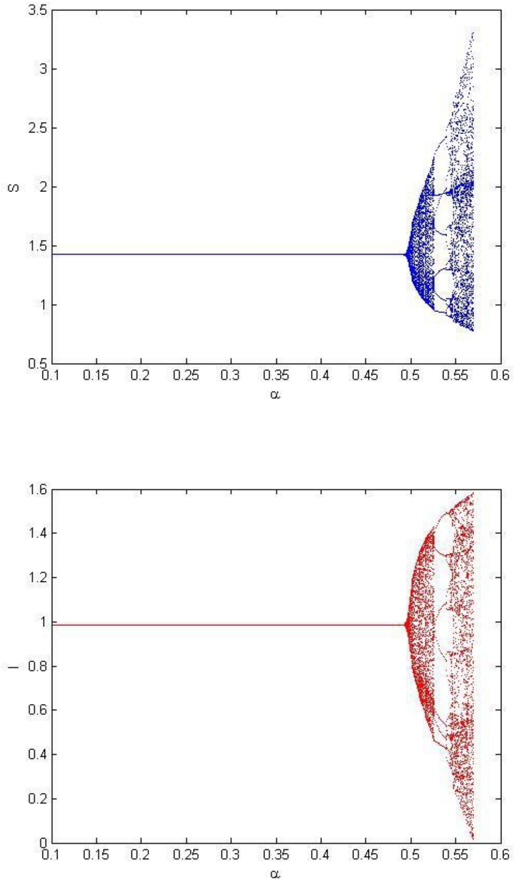
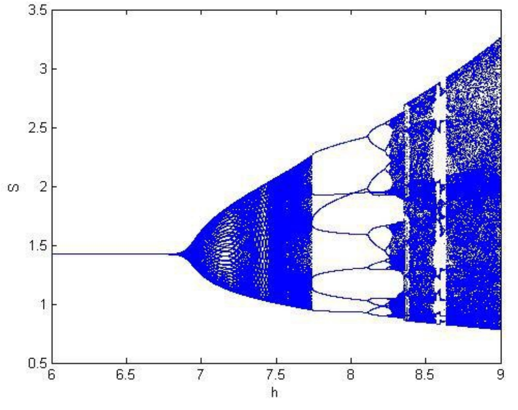


Figure 6. The bifurcation of Susceptible population, Infected population with initial values $(S_0, I_0) = (0.95, 0.05)$, $b = 1.2$, $d = 0.4$, $\beta = 0.45$, $\delta = 0.24$, $h = 7.0$ and $\alpha \in [0.1, 0.6]$.



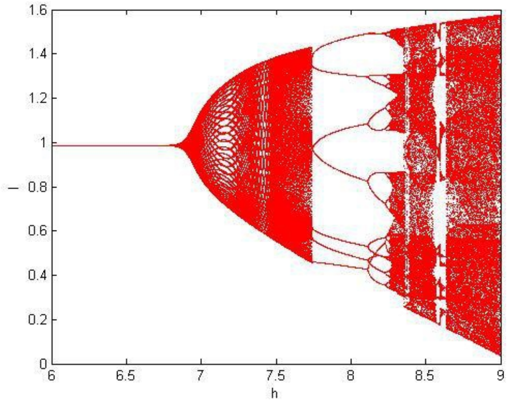


Figure 7. The bifurcation of Susceptible population, Infected population with initial values $(S_0, I_0) = (0.95, 0.05)$, $b = 1.2$, $d = 0.4$, $\beta = 0.45$, $\delta = 0.24$, $h \in [6.0, 9.0]$ and $\alpha = 0.5$.

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