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Mathematics

On ∅ 2- absorbing primary elements in Lattice Module.

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ABSTRACT In this paper we introduce the concept of ∅ -2 absorbing primary elements in lattice modules as a generalization of ∅ -2 absorbing elements and obtain some properties of such elements. The relation between ∅ -prime elements and ∅ -2 absorbing elements and also the relation between ∅ -primary elements and ∅ -2 absorbing primary is obtained. Also the characterizations of various 2-absorbing elements of lattice module are proved.

KEYWORDS : Multiplicative lattice, lattice modules, absorbing elements,prime elements, primary elements, ∅ -prime elements.

1. Introduction

A multiplicative lattice L is a complete lattice provided with commutative, associative and join distributive multiplication in which the largest element 1 acts as a multiplicative identity. An element a ∈ L is called proper if a < 1. A proper element p of L is said to be prime if ab ≤ p implies a ≤ p or b ≤ p. If a ∈ L, b ∈ L, (a : b) is the join of all elements c in L such that cb ≤ a. A proper element p of L is said to be primary if ab ≤ p implies a ≤ p or b^n ≤ p for some positive integer n. An element a ∈ L is called compact if a $\leq \vee_\alpha b_\alpha$ implies a $\leq b_{\alpha_1}$ v b_{α_2} v... v b_{α_n} for some finite subset $\{\alpha_1, \alpha_2 \ldots, \alpha_n\}$.. If a ∈ L, then $\sqrt{a}$ = v { x ∈ L_* / $x^n \leq$ a, n ∈ Z+ }. An element a ∈ L is called a radical element if a = $\sqrt{a}$. Throughout this paper, L denotes a compactly generated multiplicative lattice with 1 compact in which every finite product of compact element is compact. We shall denote by L_, the set of compact elements of L. Let M be a complete lattice and L be a multiplicative lattice. Then M is called an L- module or module over L if there is a multiplication between elements of L and M written as aB, where a ∈ L and B ∈ M which satisfies the following properties:

i) $(\vee_\alpha a_\alpha)$A= $\vee_\alpha a_\alpha$A, ∀ a_α ∈ L, A ∈ M, ii)a($\vee_\alpha A_\alpha$)= $\vee_\alpha aA_\alpha$, ∀a ∈ L , A_α ∈ M, iii)(ab)A = a(bA) ∀ a; b ∈ L, A ∈ M , iv)1B = B, v) 0B = 0_M ∀ a, a_α, b ∈ L and A, A_α ∈ M, where 1 is the supremum of L and 0 is the infimum of L. We denote by 0_M and I_M the least and the greatest element of M. Elements of L will generally be denoted by a,b,c ... and elements of M will generally be denoted by A,B,C... . Let M be an L- module. If N ∈ M and a ∈L then (N : a)= v { X ∈ M / aX ≤ N }. If A,B ∈ M, then (A : B) = v {x ∈ L= xB ≤ A}. An L- module M is called a multiplication L-module if for every element N ∈ M there exists an element a ∈ L such that N = aI_M see [3]. In this paper a lattice module M will be a multiplication lattice module, which is compactly generated with the largest element I_M compact. A proper element N of M is said to be prime if aX ≤ N implies X ≤ N or aI_M ≤ N that is a ≤ (N : I_M) for every a ∈ L,A ∈ M. If N is a prime element of M then (N : I_M) is a prime element of L, [4]. An element N < I_M in M is said to be primary if aX ≤ N implies X ≤ N or $a^n I_M$ ≤ N that is a^n ≤ (N : I_M)or some positive integer n. An element of M is called a radical if (N : I_M)= $\sqrt{N : I_M}$. If aN = 0_M implies a = 0 or N = 0_M for any a ∈ L and N ∈ M then M is called torsion free L- module.

Let $\emptyset: M \rightarrow M$ be a function on L- module M. A proper element N ∈ M is said to be ∅ -prime if for all a ∈ L, A ∈ M, aA ≤ N and aA $\nleq$ Ø (N) together imply A ≤ N or a ≤ N : I_M. A proper element N ∈ M is said to be Ø- primary if for all a ∈ L, A ∈ M, aA ≤ N and aA $\nleq$ Ø(N) together imply A ≤ N or a^n ≤ (N : I_M) for some n ∈ Z+. A proper element N ∈ M is said to be 2 potent prime if for all a ∈ L,A ∈ M, aA ≤ (N : I_M)N implies A ≤ N or a ≤ N : I_M. A proper element N ∈ M is said to be almost 2 potent prime or Ø-2 prime or simply almost prime if for all aA ≤ N, aA $\nleq$ Ø (N) = (N : I_M)N implies A ≤ N or a ≤ (N : I_M). A proper element N ∈ M is said to be almost 2 potent primary if for all a ∈ L, A ∈ M, aA ≤ (N : I_M)N implies A ≤ N or a^n ≤ (N : I_M) for some n ∈ Z+. A proper element N ∈ M is said to be 2-almost primary or Ø -2 primary or almost primary if for all aA ≤ N, aA ≤ (N : I_M)N together imply A ≤ N or a^n ≤ (N : I_M). A proper element N ∈ M is said to be n-almost primary if for all aA ≤ N, aA ≤ $(N : I_M)^{n-1}$N, (n ≥ 2) imply A ≤ N or a^n ≤ (N : I_M). A proper element N ∈ M is said to be ω- primary or ϕ_ω primary if aA ≤ N and aA $\nleq$ ϕ_ω (N) = $\wedge_{i=1}^{n}(N : I_M)^i$N imply A ≤ N or a^n ≤ (N : I_M). Sachin Ballal and V.S.Kharat have defined Ø-absorbing elements in lattice module in[15]. We generalize this concept for Ø-2 absorbing elements in lattice modules. A proper element N ∈ M is said to be Ø-2 absorbing if for a, b ∈ L, Q ∈ M, abQ ≤ N and abQ ≤ Ø (N) imply ab ≤ N : I_M or aQ ≤ N or bQ ≤ N. A proper element N ∈ M is said to be Ø-2 absorbing primary if for a, b ∈ L Q ∈ M, abQ ≤ N and abQ $\nleq$ Ø (N) implies ab ≤ N : I_M or aQ ≤ ($\sqrt{N : I_M}$)I_M or bQ ≤ ($\sqrt{N : I_M}$)I_M. Let Ø : M → M be a function. A proper element Q ∈ M is said to be n absorbing primary if a_1, a_2,…, a_n ∈ L and N ∈ M such that $a_1 a_2 \dots a_n$N ≤ Q and $a_1 a_2 \dots a_n$N $\nleq$ ϕ(Q) together imply $a_1, a_2, \dots, a_n$ ≤ (Q : I_M) or $a_1 a_2 \dots a_{i-1} a_{i+1} \dots a_n$N ≤ ($\sqrt{Q : I_M}$) I_M. A proper element N of an L-module M is said to be nilpotent if $(N : I_M)^k$N = 0M for some positive integer k. N is called idempotent if $(N : I_M)^2$N = N. An element Q ∈ M is called ω-2 absorbing primary element of M if abN ≤ Q, abN $\nleq$ $\wedge_{i=1}^{n}(Q : I_M)^i$Q implies ab ≤ (Q : I_M) or aN ≤ ($\sqrt{Q : I_M}$) I_M or bN ≤($\sqrt{Q : I_M}$) I_M. A proper element q ∈ L is called 2-absorbing primary if for every a, b, c ∈ L, abc ≤ q implies either ab ≤ q or bc ≤ $\sqrt{q}$ or ca ≤$\sqrt{q}$. A proper element Q ∈ M is called 2-absorbing primary if for every a, b ∈ L,N ∈ M such that abN ≤ Q implies either ab ≤ Q : I_M or bN ≤ ($\sqrt{Q : I_M}$) I_M or aN ≤ ($\sqrt{Q : I_M}$) I_M. Let q ∈ L be a weakly 2-absorbing primary and let a, b, c ∈ L we say (a, b, c) is triple zero of q if abc = 0, ab $\nleq$ q, bc $\nleq$ $\sqrt{q}$ and ca $\nleq$ $\sqrt{q}$. Let Q ∈ M be ϕ 2-absorbing primary element of M and a, b ∈ L,N ∈ M. If abN ≤ ϕ (Q) but ab ≤ (Q : I_M) ,bN ≤ ($\sqrt{Q : I_M}$)I_M , aN ≤ ($\sqrt{Q : I_M}$) I_M then (a, b,N) is called a Ø triple primary zero of Q. A proper element Q is said to be n- almost 2-absorbing primary element for n ≥ 2, if abN ≤ Q and abN $\nleq$ ϕ(Q) = $(Q : I_M)^n$Q implies either ab $\nleq$ (Q : I_M) or aN $\nleq$ ($\sqrt{Q : I_M}$)I_M or bN $\nleq$ ($\sqrt{Q : I_M}$) I_M.

For all these definitions and any other undefined term one can refer [2],[4],[9].

2. Ø- 2 absorbing primary elements in Lattice Module.

The study of Ø-prime and Ø-primary ideals for commutative rings is done by Bataineh M.[10], Darani A. [11]. These concepts are studied for compactly generated multiplicative lattices by CSM, AVB[12].

A study of Ø-absorbing primary elements is carried out by E.Y. Celikel, et.al.[13]. We generalize this concept for lattice modules and introduce the notion of ∅-absorbing primary elements in lattice modules. Throughout this paper ϕ denotes a function defined from M to M ∪ ∅.

Definition(2.1) Let L be a multiplicative lattice and M be a L-module. A proper element N ∈ M is said to be ∅ -2 absorbing primary if for every a, b ∈ L and Q ∈ M such that abQ ≤ N and abQ ≰ (N) imply either ab ≤ N : I_M or aQ ≤ $(\sqrt{N : I_M})\, I_M$ or bQ ≤ $(\sqrt{N : I_M})\, I_M$.

Without loss of generality we assume that ∅ (N) ≤ N.

Lemma (2.2) Let L be a multiplicative lattice and M be a L-module. A proper element N ∈ M is said to be ∅ -2 absorbing primary if for every a, b ∈ L and Q ∈ M such that abQ ≤ N and abQ ≰ (N) imply either ab ≤ N : I_M or aQ ≤ $(\sqrt{N : I_M})\, I_M$ or bQ ≤ $(\sqrt{N : I_M})\, I_M$.

Lemma 7.2.2) Let Q be a proper element of M and Ψ_1, Ψ_2 : M → M ∪ ∅ be two functions with $\Psi_1 \leq \Psi_2$. If Q is a Ψ_1 -2 absorbing primary element of M then Q is Ψ_2 -2 absorbing primary element of M.

Proof:- Let a, b ∈ L, N ∈ M such that abN ≤ Q, abN ≰ Ψ_2 (Q). Since $\Psi_1 \leq \Psi_2$, we have abN ≤ Ψ_1 (Q). As Q is Ψ_1-2 absorbing primary, we have ab ≤ Q : I_M or aQ ≤ $(\sqrt{Q : I_M})\, I_M$ or bQ ≤ $(\sqrt{Q : I_M})\, I_M$. Hence Q is Ψ_2-2 absorbing primary.

Note:- The special functions Ψ_ω : M ⟶ (M ∪ ϕ) can be defined as following. Let Q be a Ψ_ω-2 absorbing primary element of M. Then we say,

∅(Q)= ∅ ⇒ Q is a 2-absorbing primary element.

$\emptyset_0$ (Q) = 0 ⇒ Q is a weakly 2-absorbing primary element.

$\emptyset_2$ (Q)= $(Q : I_M)^2 I_M$ ⇒Q is an almost 2-absorbing primary element.

$\emptyset_n$= $(Q : I_M)^n I_M$ ⇒Q is an almost n-absorbing primary element for n > 2.

$\emptyset_\omega$ (Q)= $\wedge_{n=1}^{\infty}(Q : I_M)^n I_M$ ⇒ Q is an $\omega -$ 2-absorbing primary element.

Remark (2.3):- For any two functions Ψ_1, Ψ_2 : M → M ∪ ∅ . we say $\Psi_1 \leq \Psi_2$ if Ψ_1(A) ≤ Ψ_2(A) for each A ∈ M. Thus clearly we have the following order:- $\phi_\phi \leq \phi_0 \leq \phi_\omega \ldots \leq \phi_{n+1} \leq \phi_n \leq \ldots \leq \phi_2 \leq \phi_1$.

Theorem (2.4) Let Q be a proper element of M. Then the following statements are satisfied:- 1)Q is a 2-absorbing primary element of M ⟹ Q is weakly 2-absorbing primary element of M ⟹ Q is a ω -2 absorbing primary element of M ⟹ Q is an almost (n+1)-2-absorbing primary element of M ⟹ Q is an almost n-2-absorbing primary element of M for all n ≥2 ⟹ Q is almost 2-absorbing primary element of M. 2)Q is a ϕ -prime element of M ⟹ Q is a ϕ -2 absorbing element of M ⟹ Q is a ϕ -2 absorbing primary element of M. 3)If Q is a ϕ -primary element of M then Q is a ϕ -2 absorbing primary element of M. 4)Suppose that Q is a radical element of M. Then Q is a ϕ -2 absorbing primary element of M if and only if Q is a ϕ -2 absorbing element of M. 5)Q is an n-almost 2-absorbing primary element of M for all n ≥ 2 if and only if Q is a ω-2 absorbing primary element of M.

Proof:- 1)From the remark (2.2) we get the order $\phi_{\phi} \leq \phi_0 \leq \phi_{\omega} \ldots \leq \phi_{n+1} \leq \phi_n \leq \ \ldots \ \leq \phi_2 \leq \phi_1$.

Hence the result follows from lemma (2.3) .
2)Suppose Q is ∅-prime element of M and a, b ∈ L and N ∈ M such that abN ≤ Q and abN ≰ ϕ (Q). Then ab ≤ (Q : I_M) or N ≤ Q. So ab ≤ (Q : I_M) or aN ≤ Q and bN ≤ Q. Hence Q is ∅-2 absorbing element of M.

Suppose Q is ∅-2 absorbing element of M. Let abN ≤ Q and abN ≰ ϕ (Q). Then by definition ab ≤ Q : I_M or aN ≤ Q or bN ≤ Q which gives ab ≤ Q : I_M or aN ≤ (Q : I_M) I_M or bN ≤ (Q : I_M) I_M.

Therefore ab ≤ Q : I_M or aN ≤ ($\sqrt{Q : I_M}$) I_M or bN ≤ ($\sqrt{Q : I_M}$) I_M. Hence Q is a ∅-2 absorbing primary element of M.

3) Let abN ≤ Q where a, b ∈ L and N ∈ M and abN ≰ (Q).
Since Q is ∅-primary, we have ab ≤ ($\sqrt{Q : I_M}$) or N ≤ Q. Thus aN ≤ ($\sqrt{Q : I_M}$) I_M , bN ≤ ($\sqrt{Q : I_M}$) I_M or ab ≤ ($\sqrt{Q : I_M}$). Hence Q is ∅-2 absorbing primary.

4) Suppose Q is a radical element of M i.e. ($\sqrt{Q : I_M}$)= Q : I_M and Q is a ∅-2 absorbing primary element. Suppose abN ≤ Q and abN ≰ ϕ(Q). Since Q is ϕ -absorbing primary ab ≤ (Q : I_M) or aN ≤ ($\sqrt{Q : I_M}$) I_M = (Q : I_M) I_M= Q or bN ≤ ($\sqrt{Q : I_M}$) I_M = Q.

Hence Q is ∅-2 absorbing element. Conversely, suppose Q is a ∅-2 absorbing element of M and abN ≤ Q and abN ≰ ∅ (Q). Then we have aN ≤ Q or bN ≤ Q or ab ≤ Q : I_M. Since M is a multiplication lattice module we have Q = (Q : I_M) I_M ≤ ($\sqrt{Q : I_M}$)I_M. Thus ab ≤ (Q : I_M) or aN ≤ ($\sqrt{Q : I_M}$) I_M or bN ≤ ($\sqrt{Q : I_M}$)I_M. Thus Q is ∅-2 absorbing primary element.

5)Suppose Q is n-almost 2-absorbing primary element of M for all n ≥ 2. Let abN ≤ Q, abN ≰ $\wedge_{n=1}^{\infty}(Q : I_M)^i Q$. Then abN ≤ Q, abN ≤ $(Q : I_M)^n$Q for some n ≥ 2. Since Q is n-almost 2-absorbing primary elements, we have, ab ≤ (Q : I_M) or aN ≤ ($\sqrt{Q : I_M}$) I_M or bN ≤ ($\sqrt{Q : I_M}$) I_M. Hence Q is ω-2 absorbing primary element of M. Converse is obvious.

Theorem (2.5) Let Q be a ϕ-2 absorbing primary element of M. If ϕ(Q) is a 2- absorbing primary element of M, then Q is a 2-absorbing primary element of M.

Proof:-Let a, b ∈ L and N ∈ M such that abN ≤ Q and ab ≰ (Q : I_M). If abN ≰ ϕ(Q) then aN ≤ ($\sqrt{Q : I_M}$) I_M or bN ≤ ($\sqrt{Q : I_M}$) I_M since Q is ∅-2 absorbing primary element. So Q is a 2-absorbing primary element of M.

Suppose abN ≤ ϕ(Q). As ab ≰ (Q : I_M), abI_M ≰ Q and ϕ(Q) ≤ Q together imply abI_M ≰ ϕ (Q). As ϕ(Q) is 2-absorbing primary, ab ≰ (ϕ(Q) : I_M) we have, aN ≤ ($\sqrt{\phi(Q) : I_M}$) $I_M$or bN ≤ ($\sqrt{\phi(Q) : I_M}$) I_M. Thus aN ≤ ($\sqrt{(Q) : I_M}$) I_M or bN ≤ ($\sqrt{Q : I_M}$) I_M. Hence Q is a 2-absorbing primary element of M.

Definition (2.6) Let Q ∈ M be ϕ 2-absorbing primary element of M and a, b ∈ L,N ∈ M. If abN ≤ ϕ(Q) but ab ≰ (Q : I_M) or bN ≰ ($\sqrt{Q : I_M}$) Q or aN ≰($\sqrt{Q : I_M}$)Q then (a, b,N) is called a triple primary zero of Q.

Remark (2.7) If Q be a ϕ -2 absorbing primary element of M which is not 2-absorbing primary, then there exists (a, b,N) a ϕ -triple primary zero of Q for some a, b ∈ L and N ∈ M.

Proof:-Suppose Q is not 2-absorbing primary. Then there exist a, b ∈ L and N ∈ M such that abN ≤ Q but ab ≰ (Q : I_M) and aN ≰ ($\sqrt{Q : I_M}$) I_M and bN ≰ ($\sqrt{Q : I_M}$) I_M. Since Q is ∅-2 -absorbing primary, we must have abN ≤ ∅ (Q). Thus (a, b,N) is a ∅-triple primary zero of Q.

Lemma 7.2.8) Let Q be a ϕ -2 absorbing primary element of M and suppose (a, b,N) is a ϕ -triple primary zero of Q for some a, b ∈ L and N ∈ M, then the following hold:

1)abQ ≤ ϕ(Q), b(N : I_M)Q ≤ ϕ(Q),a(N : I_M)Q ≤ ϕ(Q). 2)a$(Q : I_M)^2 I_M$, b$(Q : I_M)^2 I_M$, N$(Q : I_M)^2$ ≤ ϕ(Q). 3) $(Q : I_M)^3 I_M$ ≤ ϕ(Q).

Proof:- 1)Assume to the contrary that abQ ≰ ϕ(Q). Then ab(N ∨ Q) ≰ ϕ (Q). Since ab(N ∨ Q) ≤ Q, ab(N □ Q) ≰ ϕ (Q),ab ≰ (Q : I_M) and Q is a ∅-2 absorbing primary, we have a(N ∨ Q) ≤ ($\sqrt{Q : I_M}$) I_M or b(N □ Q) ≤ ($\sqrt{Q : I_M}$) I_M. Thus aN ≤ ($\sqrt{Q : I_M}$) I_M or bN ≤ ($\sqrt{Q : I_M}$) I_M, a contradiction. Therefore abQ ≤ ϕ(Q). Similarly b(N : I_M) ≤ ϕ(Q) and a(N : I_M) ≤ ϕ(Q).

2)Suppose a$(Q : I_M)^2 I_M$ ≰ ϕ(Q). So a[b ∨ (Q : I_M)][N ∨ (Q : I_M) I_M] ≰ ϕ(Q), we have a[b ∨ (Q : I_M)][N ∨ (Q : I_M) I_M] ≤ Q. Since Q is a ∅-2 absorbing primary element of M, we have either a[b ∨ (Q : I_M)] ≤ (Q : I_M) or a[N ∨ (Q : I_M) I_M] ≤ ($\sqrt{Q : I_M}$) I_M or [b ∨ (Q : I_M)][N ∨ (Q : I_M) I_M] ≤ ($\sqrt{Q : I_M}$) I_M. Thus either ab ≤ (Q : I_M) or aN ≤ ($\sqrt{Q : I_M}$) I_M or bN ≤ ($\sqrt{Q : I_M}$) I_M. This is a contradiction, since (a, b,N) is a ϕ triple primary zero of Q. Hence a$(Q : I_M)^2 I_M$ ≤ ϕ (Q). Similarly b$(Q : I_M)^2 I_M$ ≤ ϕ (Q) and N$(Q : I_M)^2 I_M$ ≤ ϕ (Q).

3) Suppose $(Q : I_M)^3 I_M$ ≰ ϕ (Q). So [a ∨ (Q : I_M)][b ∨ (Q : I_M)][N ∨ (Q : I_M) I_M] ≰ ϕ (Q). We have [a ∨ (Q : I_M)][b ∨ (Q : I_M)][N ∨ (Q : I_M) I_M] ≤ Q. Since Q is ∅-2 absorbing primary, [a ∨ (Q : I_M)][b ∨ (Q : I_M)] ≤ (Q : I_M) or [a ∨ (Q : I_M)][N ∨ (Q : I_M) I_M] ≤ ($\sqrt{Q : I_M}$) I_M or [b ≤ (Q : I_M)][N ∨ (Q : I_M) I_M] ≤ ($\sqrt{Q : I_M}$) I_M. This shows that ab ≤ Q : I_M or aN ≤ ($\sqrt{Q : I_M}$) I_M or bN ≤ ($\sqrt{Q : I_M}$)I_M. This is a contradiction to the fact that (a, b,N) is a triple zero of Q. Hence a$(Q : I_M)^3 I_M$ ≤ ϕ (Q).

Corollary (2.9) Let Q be a ϕ -2 absorbing primary element of M such that ϕ ≤ $ϕ_4$. Then Q is a $ϕ_n$-2 absorbing primary element of M for every n ≥2.

Proof:-Suppose Q is a 2 absorbing primary element of M and let ϕ ≤ $ϕ_4$. Let abN ≤ Q and abN ≰ $(Q : I_M)^n I_M$. Since Q is 2-absorbing primary ab ≤ Q : I_M or aN ≤ ($\sqrt{Q : I_M}$) I_M or bN ≤

$(\sqrt{Q : I_M})\, I_M$. So Q is a Φ_n-2 absorbing primary element of M for n ≥ 2.[n ≥ 2 because (Q : I_M) = Q]

Corollary (2.10) Let Q be a proper element of M and ϕ : M → M be a function i)If Q is a ϕ -2 absorbing primary element of M such that $(Q : I_M)^3 Q \nleq \phi(Q)$ then Q is a 2- absorbing primary element of M. ii)If Q is a ϕ -2 absorbing primary element of M that is not 2 absorbing primary element of M then $(\sqrt{Q : I_M})Q = (\sqrt{\phi(Q : I_M})\phi(Q)$.

Proof:-i) Suppose Q is not ∅-2 absorbing primary element of M such that $(Q : I_M)^3 Q \nleq \phi$ (Q). If Q is not 2 absorbing primary there exists a, b,N a ∅-triple primary zero of Q for some a, b ∈ L and N ∈ M such that $(Q : I_M)^3 I_M \leq \phi$ (Q) ,by lemma (2.8) which is not true. Hence Q is 2-absorbing primary element of M.

ii) Suppose Q is a ∅-2 absorbing primary element of M that is not 2 absorbing primary. Hence by Lemma (2.8), $(Q : I_M)^3 I_M \leq \phi$ (Q). So (Q : I_M) ≤ $(\sqrt{\phi(Q) : I_M})$ implies $(\sqrt{Q : I_M}) \leq \sqrt{(\sqrt{Q : I_M})} = (\sqrt{Q : I_M})$. But ϕ(Q) ≤ Q implies $(\sqrt{\phi(Q) : I_M}) \leq (\sqrt{Q : I_M})$. Hence $(\sqrt{\phi(Q) : I_M}) = (\sqrt{Q : I_M})$.

Theorem (2.11) Let Q be a proper element of M such that $(\sqrt{\emptyset(Q) : I_M})$ is a primary element of L. If i)Q is a ϕ -2 absorbing primary element of M, then ii) Q : I_M is a 2-absorbing primary element of L.

Proof:- Suppose Q is a ∅-2 absorbing primary element of M that is not 2-absorbing primary then by corollary (2.10)(ii) $(\sqrt{Q : I_M}) = (\sqrt{\phi(Q) : I_M})$. Hence $(\sqrt{Q : I_M})$ is a primary element by hypothesis. Thus Q : I_M is a 2-absorbing primary element of L by theorem (2.7) in [14].

Theorem (2.12) Let Q be an element of a Noether lattice domain M. If Q is Φ_n-2-absorbing primary element of M for all n ≥ 2 then Q is a 2-absorbing primary element of M.
Proof:- Let abN ≤ Q for some a, b ∈ L and N ∈ M. If abN

$\nleq \Phi_n$ (Q) then as Q is Φ_n-2-absorbing primary, we have either ab ≤ (Q : I_M) or aN ≤ $(\sqrt{Q : I_M}) I_M$ or bN ≤ $(\sqrt{Q : I_M})\, I_M$. Assume that abN ≤ Φ_n (Q) for all n _ 2 then abN ≤ $\wedge_{n=1}^{\infty} \Phi_n$ (Q) = $\wedge_{n=1}^{\infty} (Q : I_M)^n I_M$ = 0Q = 0_M together imply abN = 0_M. Since M is a domain a = 0 or b = 0 or N = 0_M. So 0 = ab ≤ (Q : I_M) or bN = 0_M ≤ Q or aN = 0_M ≤ Q. [4](by corollary 3.3)

3. On 2-absorbing primary and weakly-2 absorbing elements in lattice modules:

Theorem (3.1) Let Q be a proper element of M, then i)If Q is a (weakly) prime element, then Q is (weakly) 2-absorbing primary element of M. ii)Let Q be a radical element. If Q is a (weakly) primary element, then Q is a (weakly) 2-absorbing primary element. iii)If Q is a(weakly) 2-absorbing element, then Q is a(weakly) 2-absorbing primary element.

Proof:-i)Let Q be a prime element of M. Let abN ≤ Q, a, b ∈ L,N ∈M. Since Q is prime, we have, ab ≤ Q : I_M or N ≤ Q. Therefore ab ≤ Q : I_M or aN ≤ Q and bN ≤ Q Thus ab ≤ Q : I_M or aN ≤ ($\sqrt{Q : I_M}$) I_M, and bN ≤ ($\sqrt{Q : I_M}$) I_M. Hence Q is a 2-absorbing primary element of M. Proofs of (ii) and (iii)are obvious.

Theorem (3.2) Let M be a compactly generated lattice module, i)An element Q ∈ M is a 2-absorbing primary element if and only if for any a, b ∈ L_*,N ∈ M^*, abN ≤ Q imply either ab ≤ Q : I_M or bN ≤ ($\sqrt{Q : I_M}$) I_M or aN ≤ ($\sqrt{Q : I_M}$) I_M. ii)An element Q ∈ M is a weakly 2-absorbing primary element if and only if for any a, b ∈ L,N ∈ M, 0 ≠ abN ≤ Q implies either ab ≤ Q : I_M or bN ≤ ($\sqrt{Q : I_M}$) I_M or aN ≤ ($\sqrt{Q : I_M}$) I_M.

Proof:- Assume that for a, b ∈ L_*,N ∈ M_*, abN ≤ Q imply ab ≤ Q : I_M or bN ≤ ($\sqrt{Q : I_M}$)I_M or aN ≤ ($\sqrt{Q : I_M}$) I_M. Let a, b ∈ L, abN ≤ Q and aN ≰ ($\sqrt{Q : I_M}$)I_M, bN ≰ ($\sqrt{Q : I_M}$) I_M and N ∈ M. Then there exist compact elements a' ≤ a, b'≤ b, N' ≤ N such that $a'b'N'$ ≤ Q. Since aN ≤ ($\sqrt{Q : I_M}$) I_M and bN ≤ ($\sqrt{Q : I_M}$) I_M there exist compact elements a_1 ≤ a, b_1 ≤ b, N_1 ≤ N, N_1 ≤ N such that a_1N_1 ≰ ($\sqrt{Q : I_M}$) I_M, b_1N_1 ≰ ($\sqrt{Q : I_M}$) I_M. Put $N_3 = N_1 \vee N_2 \leq N'$, $a_2 = a_1 \vee a'$, $b_2 = b_1 \vee b'$. We show that ab ≤ Q : I_M. Choose compact elements a_α ≤ a, b_α ≤ b. Then ($a_2 \vee a_\alpha$)($b_2 \vee b_\alpha$) N_3 ≤ Q implies ($a_2 \vee a_\alpha$) N_3 ≰ ($\sqrt{Q : I_M}$)Q ; ($b_2 \vee b_\alpha$) N_3 ≰ ($\sqrt{Q : I_M}$)Q and hence by hypothesis, ($a_2 \vee a_\alpha$)($b_2 \vee b_\alpha$) ≤ (Q : I_M). Hence ($a_\alpha b_\alpha$) ≤ (Q : I_M). Consequently, ab ≤ Q : I_M. Therefore Q is 2-absorbing primary element of M. The converse is obvious. ii)The proof of (ii) is same as (i).

Theorem (3.3) Q is 2-absorbing element of M then ($\sqrt{Q : I_M}$) is 2-absorbing element of L.

Proof:- Let abc ≤ ($\sqrt{Q : I_M}$) and ac ≰ ($\sqrt{Q : I_M}$)and bc ≰ ($\sqrt{Q : I_M}$). Then $(abc)^n$ ≤ Q : I_M, for some positive integer n. $a^n b^n$ ($c^n I_M$) ≤ Q. We have, $a^n c^n$ ≰ Q : I_M and $b^n c^n$ ≰ Q : I_M, $a^n c^n I_M$ ≰ Q and $b^n c^n I_M$ ≰ Q. Since Q is 2-absorbing primary element we must have $a^n b^n$ = $(ab)^n$ ≤ Q : I_M. Hence ab ≤ ($\sqrt{Q : I_M}$). So ($\sqrt{Q : I_M}$) is 2-absorbing element of L.

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