ORIGINAL RESEARCH PAPER | Mathematics | Volume - 7 | Issue - 3 | March - 2017 | ISSN - 2249-555X | IF : 4.894 | IC Value : 79.96

ON THE RELATIONSHIP BETWEEN AN ORTHOGONAL AND A COMPLETE REDUCIBLE CONTINUOUS LINEAR REPRESENTATION

KEYWORDS Topological group,bilinear function, orthogonal and completely reducible continuous linear representation

Diah Junia Eksi Palupi

Department of Mathematics, Faculty of Mathematics and Natural Sciences Universitas Gadjah Mada, Yogyakarta, Indonesia

ABSTRACT A continuous linear representation ρ_c is a homomorphism from a topological group G into GLc(V) of all continuous bijective transformations. Representation c is called an orthogonal if on the topological vectorspace V there is a positive definite symmetric bilinear function f invariant under c. Futhermore, c is said to be completely reducible if every invariant subspace U of V has an invariant complement W. We have every orthogonal representation is completely reducible. Especially for a continuous linear representation from a compact topological group.

1 Introduction

A topological vector space is a vector space that is completed by a topology and satisfies some axioms of continuous function properties [7]. A topological group is a group that is a toplogical space satisfying two axioms of continuity function [2]..The relationship among groups and vector spaces is given by a map which is called a linear representation. A linear representation of a finite group has been discussed by some researchers, such as Verlag and Pierre [8,9]. Many applications of linear representation of a finite group have been done. In 2009, Palupi et.al [4a] discussed the contruction of topology on $Lc(V)$ and in the same year, they defined a continuous linear representation from a topological group into a topological vector space [4b]. And then Palupi do more research on the continuous linear representation untill now.The topological vector space V is said to be the representation space. So, the continuous linear representation to represent an abstract space into a real space, as a complex number set or a real number set.

Futhermore, a topological vector subspace $U \subseteq V$ is said to be an invariant under a continuous linear representation c, if there is a topological vector subspace W which is an invariant under ρ_c too, such that $U \oplus W = V$.

In the otherside, we can choose a real representation space is Euclide space or Hilbert space. The continuous linear representation ρ_c is called an orthogonal if we can define a bilinear function on the Eucide or Hilbert space.

1.1 Continuous Linear Representation

Let $L_c(V)$ be a topological vector space, where $L_c(V)$ is a collection of all continuous linear operator from V into himself. Let $GL_c(V) = \{T|T: V \to V$ linear, continuous and bijective$\}$. The set $GL_c(V)$ is not empty and $GL_c(V) \subset L_c(V)$. Futhermore, by a topology GL, that is an induced by topology L on $L_c(V)$ and restriction continuous maps fL and gL of $L_c(V)$ on $GL_c(V)$, we have $GL_c(V)$ is a topological vector subspace of $L_c(V)$. Under a composition operation, $GL_c(V)$ is a topological group.

Now, let (G,μ) be a topological group. Since G is a group then for every $x,y \in G$, $xy \in G$. We can define a map P_c from (G,μ) into $GL_c(V)$ such that for every $x,y \in G$, $P_c(xy) = P_c(x)\ \ Pc(y)$. Notice that for the identity element $e \in G$, $P_c(x) = P_c(ex) = P_c(xe) = P_c(x)\ \ P_c(e) = P_c(e)\ \ P_c(x)$. Thus, $P_c(e)$ is the identity element in GLc(V). Since for each xG there is an inverse element x^{-1}G such that $xx^{-1} = x^{-1}x = e$, $P_c(xx^{-1}) = P_c(x)\ \ P_c(x^{-1}) = P_c(e) = P_c(x^{-1}x) = P_c(x^{-1})\ \ P_c(x)$, then $P_c(x^{-1}) = (P_c(x))^{-1}$. By considering the definition of $GLc(V)$, for every $x \in G$, there exists an element in $GL_c(V)$ as the image of P_c at x.

We write the image of P_c at x by T_{cx}, for every $x \in G$. That means, $P_c(x)$ will be written by Tcx, for every xG. For next discussion, P_c denotes a map from (G,μ) into $(GL_c(V),\tau_{GL})$ such that $P_c(xy) = P_c(x)\ \ P_c(y)$, for every $x,y \in G$. The map P_c has characteristic as stated in Theorem 1.1.1

Theorem 1.1.1 *Let (G,μ) and $(GL_c(V),\tau_{GL})$ be topological groups. A map $\rho c: (G,\mu)\ \ (GL_c(V),\tau_{GL})$ is a continuous homomorphism.*

Futhermore, if we have a map Pc in Theorem 2.1, we define a representation concept as follow.

Definition 1.1.2 *Let (G,μ) be a topological group and (V,σ) be a topological vector space. In Theorem 1.1.1, a map c from (G,μ) into $(GL_c(V), \tau_{GL})$ satisfies:*

(i) $\rho_c(y) = \rho_c(x) \circ \rho_c(y)$

(ii) $\rho_c(x^{-1}) = (P_c(x))^{-1}$, for every $x,y \in G$.

(iii) ρ_c is continuous

A map ρ_c state like above, is called a continuous linear representation.

The existence of this representation is a homomorfism ρc from an abritary topological group into an abritary topological vector space which is defined as $\rho_c(x) = \rho_c(e)$ for every $x \in G$ where $\rho_c(e)$ is identity element I_{cx} in $GL_c(V)$.

Let (V,σ) be a topological vector space, by considering an induced topology, every subspace U of V is a topological vector subspace such that we can write as (U, σ_U) where σ_U is a topology induced by σ. A topological vector subspace U of V is called invariant over ρc if $\rho c(x)(u) = T_x(u)$ in U, for every $x \in G$ and $u \in U$. Irreducibility and reducibility of continuous linear representation are listed in the following definition.

Definition 1.1.3 *A continuous linear representation c from a topological group (G,μ) into a topological vector space (V,σ) is called irreducible if invariant topological vector subspaces of V over c are only $\{0\}$ and (V,σ). A continuous linear representation c is called complete reducible if every invariant topological vector subspace U of V over c, there is an invariant topological vector subspace W of V over ρ_c such that $U \oplus W = V$.*

1.2 Bilinear function.

On this sub bab 1.2, we have declared the bilinear function which is simetric, positive definite and invariant under a representation. Let V_R be the real vector space or V_C be the complex vector space.

Definition 1.2.1 *Let V be an inner product space. The Function h from V × V into a field F (compleks or real) is called sesquilinear if it satisfies*

i) $h(x_1 + x_2, y) = h(x_1, y) + h(x_2, y)$

ii) $h(x, y_1 + y_2) = h(x, y_1) + h(x, y_2)$

iii) $h(\alpha x, y) = \alpha h(x, y)$, *(iv)* $h(x, \beta y) = \beta\, h(x, y)$ *for every* x_1, x_2, x, y_1, y_2, y V *and* α, βEe *F*

Futhermore, because of the axiom i) and ii), *h* is linear and linear conjugate. The function h is called bilinear if F is the real number set such that the axiom iv) gives $\beta = \beta$. The function h is called positive definite if $h(x, y) \neq 0$ for every $x, y \in V$ not zero.

On the functional analysis, we have Hilbert space and Banach space, that are vector spaces are completed by inner product and norm which are complete, every Cauchy series is convergent.

On the below, given the concept of simetric, specially the simetricity of linear operator on Hilbert space.

Definition 1.2.1 *Let H be a Hilbert space and T be a linear operator on H. The operator T is called simetric if every x, y in H,* $<Tx,y> = <x,Ty>$
The relationship between a bilinear function and representation of definition 1.1.2 , given as follows.

Definition 1.2.2 *Let (V,σ) be the topological real vector space and ρc be a continuous linear representation from a topological group (G,μ) into (V,σ). A bilinear function f on V × V is called invariant under the representation* ρ_c *iff* $f(\rho_c(g)x, \rho_c(g)y) = f(x,y)$ *for every* $g \in G$ *and* $x,y \in V$.

Futhermore, we will discuss the relationship between bilinear function and the continuous linear representation, aspecially the reducible representation.

2 Main result

From definition 1.1.2 (*V*σ) is called the representation of (G,μ). Now, let (*V*σ) be a topological Hilbert space and ρ_c be a continuous linear representation of (G, μ) has defined as action of a linear operator on a bilinear function. Hence, we can use the concept of an invariant function under a continuous linear representation, like definition 1.2.2. Let *f* be a bilinear function on V × V. An orthogonal continuous linear representation is defined as follows.

Definition 2.1 *Let (G,μ) be a topological group, (V,σ) be a topological Hilbert space and ρc be a continuous linear representation from (G,μ) into (V,σ). The representation ρc is said to be orthogonal if there is a bilinear function h which is simetric, positive definite and invariant under the representation ρc*
.

So, if we have a continuous linear representation ρc and a bilinear function *h* like above, then ρc : G → $GL_c(V)$ implies a relation *h(ρc(g)u, ρc(g)v) = h(u,v)*, for every g G and u,v V. Consider, ρc(g) is an invertible linear operator. If we choose h is an inner product function on *V × V and (V,)* be an Euclide space then $\rho_c(g)$ is an orthogonal linear operator. Hence, $h\rho_c(g) = h$ where $\rho_c(g)$ is an invertible linear operator.

If T_g denote ρ_c(g), we have $hT_g = h$. Now, we have a next theorem.

Theorem 2.2 *Let ρc be a continuous linear representation from (G,μ) into (V,σ) then maka representation c is orthogonal.*

Proof. Since ρ_c is a continuous linear representation from *(G,μ)* into (V,σ), then ρ_c is a homomorfism from G into *GLc(V)*. Let f* be a bilinear, simetric, positive definite function on *V V ×V* in relation to ρc and *f**.

We make $h = \Sigma_x \in_G f^{\wedge *}\ T_x$ where for $x \in G$, T_x is linear operator .
Hence, *h* is bilinear and simetric. Futhermore, we will show *h* is a positive definite and an invariant under ρ_c.

Because of $h(u,v) = \Sigma x \in_G) f^* \ (T_x(u), T_x(v))$, whereas *f** is a positive definite function for every $u,v \in v$, we have h is a positive definite too.

For an abritary y G, we have $hTy = \Sigma x \in Gf^*\, T_{x)} T_y = \Sigma_{x\in G} f^*\ T_x T_y = \Sigma x G f^*\ T_{xy}$ where *xyG*.So,*hTy=h*, it means *h* is an invariant under ρ_c.

Following, we discuss the relationship of an orthogonal and a complete reducible continuous linear representation c where we choose representation *(V,σ)* be topological Hilbert space. We have theorem as follows.

Theorem 2.3 *Let ρc be an orthogonal continuous linear representation from (G, μ) into(V,σ). Then, ρc is a complete reducible.*

Proof. Let U be a topological subspace of V which is an invariant under ρc. We will claim there is a topological subspace W that is an invariant under ρc , too. Such tha *U W = V.*

Since ρ_c is an orthogonal and *V* is a Hilbert space then there is a function h is defined as an inner product which is bilinear, simetric and positive definite. Beside that, because the representation ρc is orthogonal, we can take a topological subspace *W* which is orthogonal under h.

Since ρ_c(g) operator ortogonal then for an element g *G* we have ρc(g)w *W,* for every *w W. I*n the other word, *W* is a topological subspace which is invariant under ρc.
So, ρc is a complete reducible representation

We have a corollary of above theorem.
Corollary 2.4 *A continuous linear representation ρc from a topological group into topological Hilbert space is complete reducible.*

By compact topology, we have constructed a topology on a topological operator vector space $L_c(V)$ such we have a continuous linear representation ρc as we discuss before.

Let *G* be an abstract group. On *G*, we give a discrete topology such *G* is a compact topological space. The function j from *G × G* into *G,* is defined by *(x,y)* into *xy*-1 is continuos. So, *G* is a compact topological group. We have theorem as follows.

Theorem 2.5 L*et ρc be a continuous linear representation from a compact topological group (G,) into a topological vector space (V,). Then ρc a complete reducible representation.*

Proof. Consider, the concept of complete reducible and orthogonal continuous linear representation, that we discuss before, is concept for finite group. Up to isomorfism, we can talk the same concept on the compact topological group.

3 Conclusion

Let *(V,σ)* be Hilbert topological space. Then a continuous linear representation ρc from topological group *(G,μ)* into *(V,,σ)* is an orthogonal representation and futhermore it is a complete continuous linear representation.

Especially, the continuous linear representation ρc from a compact topological group *(G,μ)* into a topological vector space *(V,σ)* is a complete reducible continuous linear representation.

Acknowledgement
The authors gratefully acknowledge the support of the Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Gadjah Mada, Yogyakarta, Indonesia.

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