Original Research Paper

Volume-9 | Issue-5 | May-2019 | PRINT ISSN No 2249 - 555X

Engineering

APPLICATION OF POLYNOMIAL BASED DIFFERENTIAL QUADRATURE METHOD IN DOUBLE PHASE (OIL-WATER) FLOW PROBLEM DURING SECONDARY OIL RECOVERY PROCESS

Dr. Amit K Parikh	Mehsana Urban Institute of Sciences, Mehsana-Kherva-384012, Mehsana, Gujarat, India
Jishan K Shaikh*	Mehsana Urban Institute of Sciences, Mehsana-Kherva-384012, Mehsana, Gujarat, India. *Corresponding Author
Dr. Absar Lakdawala	Department of Mechanical, Institute of Technology, Nirma University, Sarkhej-Gandhinagar Highway Ahmedabad-382481, Gujarat, India.

ABSTRACT In this present paper, we have studied an imbibition phenomenon in the double phase (oil-water) flow through homogeneous porous media in horizontal direction. This phenomenon occurs for the duration of the secondary oil recovery process. In this paper we used Polynomial based differential quadrature method for solving nonlinear partial differential equation using uniform grid space points with the help of specific initial conditions. Numerical values and graphical presentation are given by using MATLAB.

KEYWORDS : Imbibition phenomenon, Non-linear Partial differential equation, Homogeneous Porous Media, Polynomial Based Differential Quadrature method, Uniform grid space points.

Introduction

Imbibition involves the displacement of one immiscible fluid by another. In the porous media, imbibition phenomenon is considered by increasing saturating phase. This phenomenon can be divided into two different cases: Free (spontaneous or force Free) and Forced. We shall restrict our study to the free imbibition phenomenon.

If the saturating phase is drawn into a porous medium by resources of duct pressures then this imbibition phenomenon is known as Spontaneous imbibition. It is a method of curved edges between the saturating and non-saturating phase lacking of any exterior force. Spontaneous imbibition divided into two different types due to the flow of phases: (1) Co-current imbibition (2) counter-current imbibition.

If saturating and non- saturating phases move in the same direction, then this type of Spontaneous imbibition known as Co-current imbibition while both moves in the opposite direction is known as counter-current imbibition.

This phenomenon studied by many researchers. The similarities and differences of co-current and counter-current imbibition have been studied by Pooladi-Darvish and Firoozabadi [2]. The similarity solution of the counter-current imbibition phenomenon in banded porous matrix is discussed by Yadav and Mehta [6]. Mehta [8] has discussed this phenomenon using a singular perturbation method. However, the experimental study of co-current and counter-current flows in natural porous media has been discussed by Brownscombe and Dyes [13], Blair [17], Tavassoli, Bourblaux and Kalaydjin [18]. An analytical solution of the counter-current imbibition phenomenon arising in the fluid flow through homogeneous porous media described by Joshi, Desai and Mehta [23]. Also the mathematical model and analysis of counter-current imbibition in vertical downward homogeneous porous media is discussed by Verma [24], Mehta and Verma [08, 22], Parikh, Mehta and Pradhan [26].

Statement of the problem

Consider a semi-infinite isotropic, homogeneous porous medium pouring with non- saturating fluid oil (native) in horizontal direction thoroughly bounded by solid surface excepting at one end which is referred as an imbibition surface. The imbibition confronts is presented to a neighboring arrangement of the dislodging fluid water (injected), which specially wets the medium offering ascend to counter-current imbibition that is unconstrained rectilinear pour of saturating fluid into the medium and a counter pour of the non- saturating fluid from the medium. In counter-current flow, oil and water flow in reverse direction and oil escapes by flowing back along a similar direction along which water has assimilated. The displacement of oil takes place through the same boundary from which the saturating fluids enters the porous matrix because the other boundaries are being close.

Basic assumptions

Assume that two immiscible fluids occupy the entire pore space, that is, the sum of their saturations equal one. One fluid is saturating (water) and the other one is non-saturating (oil). We neglect gravity as well as the compressibility of the fluids and of the porous medium.

The imbibition phenomenon is considered in a homogeneous, isotropic and incompressible porous medium with finite in extent in a horizontal direction. The counter current flow of water and oil is considered to be linear and laminar in order to express the seepage velocity of fluids by Darcy's law. The standard relationships are followed by duct pressure and comparative porousness as a function of saturation of water. It is further assumed that there is no internal force or sink of either oil or water.

Mathematical Modelling of the Imbibition Phenomenon:

In the mathematical modelling of the counter current imbibition phenomenon,we have assumed that the flow is one dimensioanl and also we assumed that governed by Darcy's law. The seepage velocities of fluids are given as, [4]

$$V_i = -\frac{k_i}{\mu_i} K \frac{\partial P_i}{\partial x} \qquad (1)$$

$$V_n = -\frac{k_n}{\mu_n} K \frac{\partial P_n}{\partial x} \qquad (2)$$

where

K is the permeability of the porous medium

k_i = Relative permeability of injected fluid

k_n = Relative permeability of native fluid

P_i = Pressure of injected fluid.

P_n = Pressure of native fluid.

μ_i = density of injected fluid.

μ_n = density of native fluid.

S_i = Saturation of injected fluid.

S_n = Saturation of native fluid.

The relationship between capillary pressure, native pressure and injected pressure is given as [5]

$$P_c = P_n - P_i \tag{3}$$

Now we partially differentiate equation (3) with respect to x, so equation (3) can be written as

$$\frac{\partial P_c}{\partial x} = \frac{\partial P_n}{\partial x} - \frac{\partial P_i}{\partial x}$$

$$\frac{\partial P_n}{\partial x} = \frac{\partial P_c}{\partial x} + \frac{\partial P_i}{\partial x} \tag{4}$$

The analytic condition of governing counter current imbibition phenomenon [6] is,

$$V_i = -V_n \tag{5}$$

From equation (1), (2) and (5)

$$-\frac{k_i}{\mu_i} K \frac{\partial P_i}{\partial x} = \frac{k_n}{\mu_n} K \frac{\partial P_n}{\partial x}$$

$$\frac{\partial P_i}{\partial x} = -\left\{ \frac{K\left(\frac{k_n}{\mu_n}\right)}{\left(\frac{k_i}{\mu_i} + \frac{k_n}{\mu_n}\right)} \right\} \frac{\partial P_c}{\partial x} \tag{6}$$

Replacing the value of $\frac{\partial P_i}{\partial x}$ from equation (6) in equation (1) we get,

$$V_i = \left\{ \frac{K\left(\frac{k_n}{\mu_n}\right)\left(\frac{k_i}{\mu_i}\right)}{\left(\frac{k_i}{\mu_i} + \frac{k_n}{\mu_n}\right)} \right\} \frac{\partial P_c}{\partial x} \tag{7}$$

Neglecting the phase density variation, the equation of continuity can be written as,

$$\phi \frac{\partial S_i}{\partial t} + \frac{\partial V_i}{\partial x} = 0 \tag{8}$$

$$\phi \frac{\partial S_n}{\partial t} + \frac{\partial V_n}{\partial x} = 0 \tag{9}$$

Replacing the value of V_i from equation (7) the continuity equation (8) can be written as,

$$\phi \frac{\partial S_i}{\partial t} + K \frac{\partial}{\partial x}\left[\left\{ \frac{\left(\frac{k_n}{\mu_n}\right)\left(\frac{k_i}{\mu_i}\right)}{\left(\frac{k_i}{\mu_i} + \frac{k_n}{\mu_n}\right)} \right\} \frac{\partial P_c}{\partial x} \right] = 0 \tag{10}$$

The capillary pressure gradient in equation (10) is now given by,

$$\frac{\partial P_c}{\partial x} = \frac{dP_c}{dS_i} \frac{\partial S_i}{\partial x} \tag{11}$$

Using the above relation, so equation (10) may be written as,

$$\phi \frac{\partial S_i}{\partial t} + K \frac{\partial}{\partial x}\left[\left\{ \frac{\left(\frac{k_n}{\mu_n}\right)\left(\frac{k_i}{\mu_i}\right)}{\left(\frac{k_i}{\mu_i} + \frac{k_n}{\mu_n}\right)} \right\} \left(\frac{dP_c}{dS_i}\right)\left(\frac{\partial S_i}{\partial x}\right) \right] = 0 \tag{12}$$

Scheidegger [7] suggested the approximation as;

$$\frac{\left(\frac{k_n}{\mu_n}\right)\left(\frac{k_i}{\mu_i}\right)}{\left(\frac{k_i}{\mu_i}+\frac{k_n}{\mu_n}\right)} \approx \frac{k_n}{\mu_n} \tag{13}$$

Thus considering equation (12) for gravity free capillary imbibition of a saturating fluid in the presence of non- saturating fluid and using equation (13), it may be written as,

$$\frac{\partial S_i}{\partial t} = -\frac{\partial}{\partial x}\left\{\left(\frac{K}{\phi}\right)\left(\frac{k_n}{\mu_n}\right)\frac{dP_c}{dS_i}\frac{\partial S_i}{\partial x}\right\} \tag{14}$$

Mehta [8] assumed the following specific relationship,

$$P_c\left(S_i\right) = -\beta S_i \quad and\ k_n = 1 - S_i \tag{15}$$

Where $\beta = \sigma\sqrt{\frac{\phi}{K}}\ \left(N/m^2\right)$ is the capillary pressure parameter, σ is the interfacial tension between native and injected fluid. Using equation (15) in equation (14), we have,

$$\frac{\partial S_i}{\partial t} = \frac{\beta K}{\phi\mu_n}\frac{\partial}{\partial x}\left\{\left(1-S_i\right)\frac{\partial S_i}{\partial x}\right\} \tag{16}$$

We introduce the dimensionless numbers $X = \frac{x}{L}$ and $T = \frac{K\beta}{\phi L^2 \mu_n}t = \frac{\sigma\sqrt{K}}{\sqrt{\phi}\mu_n L^2}t$ Here the dimensionless time is similar to the dimensionless time defined by Mattax and Kyte (1962).

Using above dimensionless parameters and replacing $1 - S_i = S$ in equation (16) reduces to,

$$\frac{\partial S}{\partial T} = \frac{\partial}{\partial X}\left(S\frac{\partial S}{\partial X}\right) \tag{17}$$

Above equation (17) is the governing equation.

Introduction to Differential Quadrature Method

The Differential Quadrature method is very powerful and efficient method to obtain the solution of linear and non-linear partial differential equations as compared to other conventional methods. This method requires less computer storage as well as time [10]. Bellman et al in 1972 introduced this method.

In this method there is a basic procedure to determine the weighting coefficients. Richard Bellman used two different approaches to obtain the weighting coefficients for the first and second order partial derivative. For the first method, a simple function used as test functions but as he used large no of sampling points, the coefficient matrix becomes ill. So due to this reason he produced second method. This second method has similarity with the first method except coordinates of grid points which should be chosen as the roots of the *thN* order Legendre polynomial. After the efforts of Bellman, Chang Shu had given a remarkable influence to generalize the idea of the Polynomial based differential quadrature method.

The present problem is solved by using Polynomial based DQM [10, 16]. In this method first we have to obtain first order and second order weighting coefficients. Here in this procedure we choose Lagrange polynomial.

Let us consider *N* is total number of grid points in the given interval with coordinates as $0 = x_1, x_2, x_3, ..., x_n = 1$ For the accuracy, selection of sample points plays a dynamic role. Solving this present problem we selected uniform grid point system. That means the grid point has same step size [10].

i.e. $\Delta x = x_k - x_{k-1}, where\ k = 1,2,3,...,n$ (18)

According to Bellman let us consider $S(x)$ is suitably smooth over the given interval [0, 1], then the first order derivative $S^{(1)}(x_i)$ approximated at any grid point as

$$S_x^1(x_i) = \sum_{j=1}^{N} a_{ij}\,S\left(x_j\right),\ for\ j = 1,2,3...,N \tag{19}$$

By using Bellman's first and second approaches, Shu generalized the weighting coefficients as follows.

The off-diagonal terms of the weighting coefficient matrix of the first derivative are given by [17].

$$a_{ij} = \frac{L^{(1)}\left(x_i\right)}{\left(x_i - x_j\right)L^{(1)}\left(x_j\right)},\ i \neq j,\ i,\ j = 1,2,3...,N \tag{20}$$

Where

$$L^{(1)}\left(x_i\right) = \prod_{i=1, i\neq k}^{N}\left(x_i - x_k\right),\ i,k = 1,2,3...N \tag{21}$$

$$L^{(1)}\left(x_j\right) = \prod_{j=1,\ j\neq k}^{N}\left(x_j - x_k\right),\ \ j,k = 1,2,3...N \tag{22}$$

And diagonal terms are given by

$$a_{ii} = -\sum_{j=1}^{N} a_{ij}\ ,\ \ i = 1,2,3....,N \tag{23}$$

Equations (20) and (23) are two formulations to compute the weighting coefficients for the first order derivative. For the second order derivative, we can introduce similar approximation form [10].

$$S_x^2(x_i)=\sum_{j=1}^{N} b_{ij}\, S\left(x_j\right),\ for\ \ j=1,2,3...,N \qquad (24)$$

Where $S_x^{\,2}(x_i)$ is the second order derivative of $S(x)$ at x_i, b_{ij} is the weighting coefficient of the second order derivative expressed as [10].

$$b_{ij}=2\,a_{ij}\left[a_{ii}-\frac{1}{x_i-x_j}\right], for\, i\neq j \qquad (25)$$

And

$$b_{ii}=-\sum_{j=1,\, j\neq i}^{N} b_{ij} \qquad (26)$$

b_{ij} is calculated from equation (25) and b_{ii} from equation (26).

Application of DIFFERENTIAL QUADRATURE METHOD in Imbibition

To solve the governing equation (17) of the Imbibition phenomenon, we are using differential quadrature method.
The governing equation (17) can be rewritten as

$$\frac{\partial S}{\partial T}=\left(\frac{\partial S}{\partial X}\right)^2+S\left(\frac{\partial^2 S}{\partial X^2}\right) \qquad (27)$$

Now discretizing the governing equation (27) using the following expression (28) to (30)

$$\frac{\partial S}{\partial T}=\frac{dS_i}{dT},\ i=1,2,3,...,N \qquad (28)$$

Where $S_i=S(x_i)$
i.e. the saturation of injected fluid at *ix*

$$\frac{\partial^2 S}{\partial X^2}=\sum_{j=1}^{N} b_{ij}\, S_j\,,\quad j=1,2,3,...,N \qquad (29)$$

$$\frac{\partial S}{\partial X}=\sum_{j=1}^{N} a_{ij} S_j\,,\quad j=1,2,3,...,N \qquad (30)$$

Where $S_j=S(x_j)$
i.e. the saturation of injected fluid at x_j
Replacing the first and second order partial derivative in equation (27) by (28) to (30), we get the systems of the first order ODE in the form.

$$\frac{dS_i}{dT}=\left(\sum_{j=1}^{N} a_{ij} S_j\right)^2+S_i\left(\sum_{j=1}^{N} b_{ij}\, S_j\right) \qquad (31)$$

Consider the initial condition as $S(X,0)=1-X^2,\ 0<X<1$ (32)
Here solving this problem, we are using uniform grid points. The weighting coefficients a_{ij} and b_{ij} of first and second order derivatives can be obtained by using the formula given in equation (20) and (25) respectively.

The weighting coefficients and for grid points N=5 are:

$a_{11}=-8.33333,\ a_{12}=-16,\ a_{13}=12,\ a_{14}=-5.33333,\ a_{15}=1$
$a_{21}=1,\ a_{22}=-3.3333,\ a_{23}=-6,\ a_{24}=2,\ a_{25}=-0.3333$
$a_{31}=-0.3333,\ a_{22}=2.66667,\ a_{33}=0,\ a_{34}=-2.66667,\ a_{35}=0.3333$
$a_{41}=0.33333,\ a_{42}=-2,\ a_{43}=6,\ a_{44}=3.33333,\ a_{45}=-1$
$a_{51}=-1,\ a_{52}=5.3333,\ a_{53}=-12,\ a_{54}=16,\ a_{55}=8.3333$

Similarly,

$b_{11}=46.66667,\ b_{12}=138.6667,\ b_{13}=-152,\ b_{14}=74.66667,\ b_{15}=-14.6666$
$b_{21}=-14.66667,\ b_{22}=-26.66667,\ b_{23}=-8,\ b_{24}=-5.33333,\ b_{25}=1.3333$
$b_{31}=1.3333,\ b_{32}=-21.3333,\ b_{33}=-40,\ b_{34}=-21.3333,\ b_{35}=1.3333$
$b_{41}=1.33333,\ b_{42}=-5.33333,\ b_{43}=-8,\ b_{44}=-26.6667,\ b_{45}=-14.6667$

Now, using appropriate initial condition, $S(X,0)=1-X^2,\ 0<X<1$ we will obtain $S_1=S\left(X_1,0\right)=S\left(0,0\right)=1$
Similarly, S_2=0.9375, S_3=*0.75*, S_4=0.4375, S_5=*0.*

Using above values and Weighting coefficients in equation (31) we will get following the system of first order differential equations

$$\frac{dS_1}{dT}=0.999999781+0.0000001875\,S_1 \qquad (33)$$

$$\frac{dS_2}{dT}=5.640624629-10.99999997\,S_2 \qquad (34)$$

$$\frac{dS_3}{dT}=0.062500016-26.99999981\,S_3 \qquad (35)$$

$$\frac{dS_4}{dT}=8.265624371-13.00000003\,S_4 \qquad (36)$$

$$\frac{dS_5}{dT}=3.999999688+6.000000187\,S_5 \qquad (37)$$

Using RK4 method for solving the above system of equations with uniform step value h in the time trend.

Table: (1a) Numerical results of Saturation in Porous media for N=5

X	0	0.25	0.5	0.75	1
T					
0	0.652142	0.690242	0.729996	0.764992	0.795422
0.1	0.558952	0.601246	0.635311	0.674612	0.710122
0.2	0.460943	0.500124	0.541203	0.594124	0.643224
0.3	0.370635	0.414332	0.461201	0.505699	0.564252
0.4	0.307834	0.351992	0.396122	0.456340	0.508514
0.5	0.248958	0.289837	0.330192	0.399744	0.461255
0.6	0.182468	0.230454	0.279452	0.348819	0.422313
0.7	0.127046	0.181952	0.237680	0.308487	0.381132
0.8	0.081176	0.140135	0.199422	0.267110	0.335261
0.9	0.041242	0.104618	0.167800	0.232821	0.296838

Table (2a) Numerical results of Saturation in Porous media for N=7

X	0	0.166667	0.333333	0.5	0.666667	0.833333	1
T							
0	0.755856	0.771201	0.7857240	0.8070885	0.8260474	0.8412085	0.8709058
0.1	0.642719	0.661234	0.6803108	0.7092745	0.7182382	0.7372019	0.7591535
0.2	0.530943	0.558950	0.5762622	0.5846934	0.6031246	0.6215558	0.6427974
0.3	0.420635	0.464332	0.4947260	0.512612	0.5367522	0.5660892	0.5866226
0.4	0.327834	0.361992	0.3961225	0.4200253	0.4538350	0.4985140	0.5280688
0.5	0.248958	0.289837	0.3419216	0.370546	0.4109011	0.4512556	0.4850616
0.6	0.182468	0.230454	0.2794528	0.327073	0.3646932	0.4123134	0.4558072
0.7	0.127046	0.181952	0.2376806	0.2921645	0.3366484	0.3811323	0.4277750
0.8	0.081176	0.140135	0.1994227	0.2580355	0.3066483	0.3552611	0.40134115
0.9	0.041240	0.104618	0.1678009	0.2308135	0.2738261	0.3268387	0.3708555
1	0.017421	0.080960	0.1434786	0.207003	0.2505274	0.3040518	0.3418751

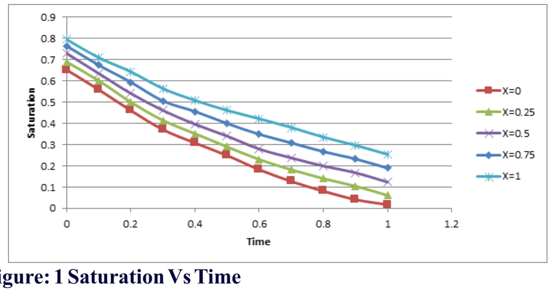

Figure: 1 Saturation Vs Time

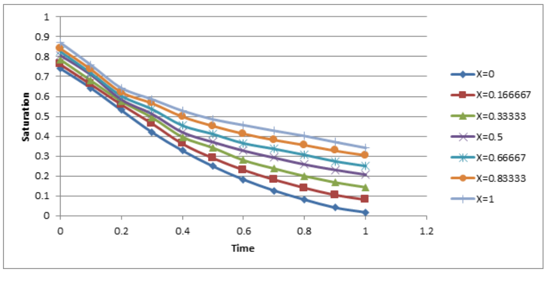

Figure: 2 Saturation Vs Time

Conclusion:

In the present study, we have applied Polynomial based Differential quadrature method on double phase (Oil-Water) flow problem during secondary oil recovery process and solved the governing nonlinear partial differential equation(27) of Imbibition Phenomenon.

In this method, we have considered Bellman second approach to obtain weighting coefficient which leads to the system of first order differential equations (32) to (36). Then using Runge-Kutta 4th order method, we have obtained the numerical solution for the Saturation of water, which is given in table (1a), and (2a). The solution is also presented graphically in the figure (1) and (2) from the numerical and graphical solution by using MATLAB. We can observe that the saturation of water is decreasing with respect to time and increasing with respect to the distance which is consistent with the physical nature of the phenomenon.

Acknowledgment:

I presented this paper in 32nd International Conference on INDO-CANADIAN multidisciplinary research: Trends and Prospects Institute of Infrastructure Technology Research and Management (IITRAM) Ahmedabad, Gujarat 28/12/2018 to 29/12/ 2018.

REFERENCES

[1] Mason G., Fischer H., Morrow N.R., Ruth D.W. Correlation for the effect of fluid viscosity on counter-current spontaneous imbibition, The Journal of petroleum science and engineering, 72(2010) 195-205.

[2] Polladi-Darvish M., Firoozabadi A., Cocurrent and Countercurrent imbibition in a water wet matrix block, Society of Petroleum Engineers Journal, Vol. 5(1), 3-11, (2000).

[3] Zhang Y, Han D, Wu K, Rao H, Imbibition oil recovery theory and influencing factors, Research Institute of Petroleum Exploration and development Postdoctoral station report(2011) 1-13.

[4] Ryzhik V., On capillary imbibition by water of an oil-saturated reservoir, Sov. Acad. Izv.: Mech. Gas Fluids 2 (1960) 149-151.

[5] Bear J. (1972). Dynamics of fluids in Porous media. American Elsevier Publishing Company, Inc.

[6] Yadav S., Mehta M.N., Mathematical model and similarity solution of countercurrent imbibition phenomenon in banded Porous matrix, Int. J of Appl. Math and Mech., Vol 5(5), 76-86, (2009).

[7] Scheideggar A.E., Johnson E.F. , The statistical behavior of Instabilities in displacement process in porous media, Canadian Journal of Physics 39 (1961) 326-333.

[8] Mehta M.N, Asymptotic expansions of fluid flow through porous media, Ph. D. Thesis, South Gujarat University, Surat, (1977).

[9] Desai K. R., Pradhan V.H., Daga A.R., Mistry P R, Approximate Analytical Solution of Non-Linear Equation in One Dimensional Imbibitions Phenomenon in Homogeneous Porous Media by Homotopy Perturbation Method. Elsevier, ScienceDirect Procedia Engineering 127(2015), 994-1001.

[10] Shu Chang (2000). Differential Quadrature method and its application in Engineering. Springer, Great Britain, 1-174.

[11] Bellman R.E.,Casti J.(1971), " Differential quadrature and long-term integration". Journal of mathematical analysis and Applications 34, 235-238.

[12] Bellman R.E; Kashef, B.G; Casti, J(1972). "Differential Quadrature: A technique for the rapid solution of nonlinear partial differential equations", J. Comput. Phys. 10, 40-52.

[13] E.R Brownscombe, A.B. Dyes, A water imbibition Displacement- A possibility for the Spraberry, drilling and production practice, American Petroleum Institute, 383-390(1952).

[14] Shu Chang.; Richards B.E. (1992a), "Application of generalized differential quadrature to solve two-dimensional incompressible Navier-stokes Equations", international J. Numer. Methods Fluids, 15, 791-798.

[15] Kashef B.G.; Bellman R.E.;(1974), "Solution of the partial differential equation of the Hodgkings-Huxley model using differential quadrature", Math. Biosci. 19, 1-8.

[16] Malik M ; Bert C.W.; Kulkretic A. R. (1993), "Differential quadrature solution of uniformly loaded circular plate resting on elastic half-space", in Aliabadi, M.H. and Brebbia, C.A. (eds), Contact Mechanics: Computational Technique, 385-396, Computational Mechanics Publicatons, Southampton, UK.

[17] P.M. Blair, Calculation of oil displacement by countercurrent water imbibition, Society of petroleum Engineers Journal, Vol 4(3), 195-202, (1964).

[18] Bourblaux J., Kalaydjian F.J., Experimental Study of Cocurrent and Countercurrent flows in naturally porous media, SPE reservoir Engineering Soc. Of Petroleum Engineers. Vol 5(3). 361-368 (1990)

[19] Z. Tavassoli, R.W. Zimmerman, M. J. Blunt, Analysis of countercurrent imbibition with gravity in weakly water-wet systems, J. Petro. Sci. and Engg. Vol 48, 94-104, (2005).

[20] Graham J.W., Richardson, J.G. (1959). Theory and applications of imbibition phenomena in recovery of oil. Trans, AIME, 216.

[21] Scheideggar A.E. (1960) The physics of flow through porous media, University of Toronto Press, 229-231,216,224.

[22] Verma A.P., Mohan Ram (1979). Existence and Uniqueness of similarity solutions of imbibition equation. Proc. Indian Acad. Soc. 88A, 3, 409-417.

[23] Joshi M.S., Desai N.B., M.N. Mehta, An analytical solution of countercurrent imbibition phenomenon arising in fluid flow through homogeneous porous media, British J. of Math and Comp Sci., Vol 3(4) 478-489 (2013).

[24] Pradhan V.H., Verma, A.P. (1977). On imbibition equation in the theory of fluid flow through porous media. A numerical approach, J. Indian Acad. Math. 19,1,19-29.
[25] Maher R, Mehta, M.N. and Maher S. (2011). Exponential self-similar solutions technique for imbibition phenomenon arising in double phase flow through porous medium with capillary pressure. Int. J. of Appl. Math. And Mech. 7(8) 29-40, 2011.
[26] Parikh A K. (2013). Generalized Separable Solutions of Counter current imbibitions phenomenon in homogeneous porous medium in horizontal direction. Int. J. of Engg. And Science. (IJES) Vol. 2, 10 pp 220-226 (2013).