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Mathematics

A STUDY OF IDENTITIES INVOLVING GENERALIZATION OF H-FUNCTIONS

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ABSTRACT The H-functions, introduced by Fox in 1961, are special functions of a very general nature, which allow one to treat several concepts including generalization of H-functions in two and multi-variables. In the present paper, the author has introduced some identities involving generalization of H-functions. The main objective of the present paper is to establish some particular and special cases of identities involving generalization of H-functions of two variables.

KEYWORDS : Fox H-Functions, Mellin-Barenes integrals, Identities, Hyper-geometric Function.

INTRODUCTION:

The *H* functions, introduced by Fox **[1]** in 1961 as symmetrical Fourier kernels, can be regarded as the extreme generalization of the generalized hyper-geometric functions *pFq* , beyond the Meijer *G* functions. Like the Meijer *G* functions, the Fox *H* functions turn out to be related to the Mellin–Barnes integrals and to the Mellin transforms, but in a more general way. After Fox, the *H* functions were carefully investigated by Braaksma **[2]**, who provided their convergent and asymptotic expansions in the complex plane, based on their Mellin–Barnes integral representation.

The Mellin – Barnes Integral

The Mellin Barnes integral is an inverse Mellin integral involving Gamma functions in the integrand. Most generally it is of the form

$$\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\alpha_1+A_1s)\ldots\ldots\ldots\Gamma(\alpha_1+A_ns)}{\Gamma(\gamma_1+C_1s)\ldots\ldots\ldots\Gamma(\gamma_n+C_nS)}\times\frac{\Gamma(\beta_1+B_1s)\ldots\ldots\ldots\Gamma(\beta_1+B_ns)}{\Gamma(\beta_1+D_1s)\ldots\ldots\ldots\Gamma(\beta_{n}+D_nS)} \quad (1.2.1)$$

Where the poles of $\Gamma(\alpha_i+A_is)$ can be separated from the poles of $\Gamma(\beta_j-B_js)$ and the integration contour is taken to the right of the $\Gamma(\alpha_i+A_is)$ and to the left of the $\Gamma(\beta_j-B_js)$ poles in the common strip of analycity. Integrals of this type lead to a variety of special functions. Often times the conditions of separation of the poles can be relaxed to a more general case in which the poles on the left overlap the poles on the right. In this case the integration contour is taken along the imaginary axis, and appropriately indented to separate the left and right hand poles. Often times these integrals can be evaluated by straightforward applications of Parseval's formula and the theory of residues to pick up the poles in one half of the plane. One function which can be useful in evaluating certain Mellin Barnes integral is the Beta function given by the integral

$$B(x,y)=\int_0^1 t^{x-1}(1-x)^{y-1}dt \quad (1.2.2)$$

The Beta function also has a very useful representation in terms of the Γ functions,

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (1.2.3)$$

Using Beta functions, we can write the following two useful Mellin Transform.

Consider the function $x^a(1-x)^{b-a-1}u(1-x)$, *u(x)* is the Heaviside function,

$$\mathcal{M}\{x^a(1-x)^{b-a-1}u(1-x)\}=\int_0^1 x^{a+s-1}(1-x)^{b-a-1}dx$$

$$=\frac{\Gamma(a+s)\Gamma(b-a)}{\Gamma(b+s)} \quad (1.2.4)$$

And similarly consider the function $x^a(1+x)^{-b-a}$, then

$$\mathcal{M}\{x^a(1+x)^{-b-a}\}=\int_0^\infty x^{a+s-1}(1+x)^{-(a+s-1)-(b+1-s)}dx$$

$$=\frac{\Gamma(a+s-1)\Gamma(b+1-s)}{\Gamma(b+a)} \quad (1.2.5)$$

H-Functions (Special Function)

The differential equations have solutions which can be expressed in terms of known functions, but the differential equation.

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+(m^2-\frac{n^2}{x^2})y=0 \quad (1.3.1)$$

Can't be solved in terms of known elementary functions. Hence in this case we define new functions in terms of the solutions (1.3.1). we call equation (1.3.1) as the Bessel's equation and the solution of this equation is known Bessel's function.

The problems of the mathematical physics lead us to determine the solutions of differential equations which satisfy certain prescribed conditions. Frequently, these solutions turn out to be new function possessing interesting properties and there are many differential equations of this type. Thus the theory of differential equations has originated a large number of new functions. We call them as "special functions".

An equation of the form

$P_0(x)y^n+P_1(x)y^{n-1}+\ldots+P_n(x)=0$ (1.3.2)

Where $P_0(x)$, $P_1(x)$, … , $P_n(x)$ are polynomial expressing having integral coefficient , e,g, function of x or constant and do not contain y, is called algebraic equation. Let the root of the above equation is as follows:

$y=f(x)$ (1.3.3)

are called algebraic function, the functions which are not root of the algebraic equation are called transcendental functions. Transcendental function such as Beta function , Gamma function , Jacobi polynomial function , Lagurerre polynomial generating function and H-function of one or more variables which are of complicated nature are known as higher transcendental function. Since a great many of the special function that occurs rather frequently in problem of applied mathematical analysis in terms of **H-function.**

More recently, the H function , being related to the Mellin transform, have been recognized to play a fundamental role in the probability theory and in fractional calculus as well as in their application including non-Gaussian stochastic processes and phenomena of non-standard.

Fox, C (1961) introduced a more general function which is well known in literature as Fox's H function or the H functions. This function is defined and represented by means of the Mellin Barnes type of contour integral

$$H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j,\alpha_j)\,1,p\\(b_j,\beta_j)\,1,q\end{matrix}\right.\right]=H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,\alpha_1)\\(b_1,\beta_1)\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi\omega}\int\theta(s)z^s ds \quad (1.3.4)$$

Where $=\sqrt{-1}$, $(z\neq 0)$ is a complex variable.

The important of the study of fox's H-function lies in the fact that all the special functions developed for the H functions Goyal, S.P, Gupta, K.C and Srivastavaa, H.M [1982]**[3]** become a master or key formula from which a considerably large number of relations for other special function can be detected merely by suitably specializing the parameters of the H function involved.

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The first systematic study of the H functions was made by Fox in the paper referred to earlier for the case where some special relations between α_j, β_j, a_j, b_j are satisfied. Fox also derived theorems about the H function as symmetrical Fourier kernels and a theorem about the asymptotic behavior of this function for $x\to\infty$ and $x>0$, Braaksma (1963)**[2]** has given a detailed account of the asymptotic expansions and analytic continuation for the general H function in a systematic manure α_j, β_j, a_j, b_j known memoir Braaksma (1963). Gupta, K.C (1965)**[4]** has established certain operational properties for this function. Gupta, K.C and Jain, U.C**[5]** have evaluated an integral involving the product of two H-functions. Gupta, K.C and Jain, U.C (1966)**[5]** on account of the useful, general character and popularity of the H-functions, a number of research workers are engaged at present in the study and further development of this function.

According to a standard notation, the Fox H function is defined as

$$H^{m,n}_{p,q}(z) = 1/2\pi i \int_{\mathcal{L}} \mathbb{H}^{m,n}_{p,q}(s)\, z^s \, ds \qquad (1.3.5)$$

Where $\mathcal{L}$ is a suitable path in the complex plane ω to be disposed later,

$z^s = \exp\{s(\log|z| + i\arg z)\}$ and

$$\mathbb{H}^{m,n}_{p,q}(s) = \frac{A(s)B(s)}{C(s)D(s)}$$

Where, $A(s) = \prod_{j=1}^{m} \Gamma(b_j - \beta_j s)$

$B(s) = \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)$

$C(s) = \prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s)$

$D(s) = \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s)$

With $0 \le n \le p$, $1 \le m \le q$, $\{a_j, b_j\} \in \mathbb{C}$, $\{\alpha_i, \beta_j\} \in R^+$

An empty product, when it occurs, is taken to be one so

$n=0 \Leftrightarrow B(s) = 1$

$m=q \Leftrightarrow C(s) = 1$

$n=p \Leftrightarrow D(s) = 1$

the above integral representation of the H functions by involving products and ratios of Gamma functions, is known to be Mellin Barnes integral type.

A compact notation is usually adopted for (1.3.5)

$$H^{m,n}_{p,q}(z) = H^{m,n}_{p,q}\left[z\Big|^{(a_1,\alpha_1)}_{(b_1,\beta_1)}\right]$$

Thus, the singular points of the kernel H are the poles of the Gamma functions entering the expressions of A(s) and B(s), that we assume do not coincide. Denoting by p (A) and p (B). the sets of these poles, we write $p(A) \cap p(B) = \emptyset$. The conditions for the existence of the H functions can be made by inspecting the convergence of integral (1.3.5) which can depend on the selection of the contour $\mathcal{L}$ and on a certain relations between the parameters $\{a_1, \alpha_1\}$ and $\{b_1, \beta_1\}$.

IDENTITIES INVOLVING GENERALIZATION OF H-FUNCTIONS OF TWO VARIABLES

IDENTITY-I

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}\, H^{M+m,\ N\ :\ M_1,N_1;\ M_2,N_2}_{P+m,\ Q+m\ :\ P_1,Q_1;\ P_2,Q_2}\left[\begin{matrix}x\\ y\end{matrix}\Big|\begin{matrix}{}_1(a_j;\alpha_j,A_j)_P\,,\,{}_1(\delta_i+r;\eta'_i,\eta''_i)_m:\ {}_1(c_j;C_j)_{P_1};\,{}_1(e_j,E_j)_{P_2}\\ {}_1(v_i+r;\mu'_i,\mu''_i)_m\,,\,{}_1(b_j;\beta_j,B_j)_Q:\ {}_1(d_j;D_j)_{Q_1};\,{}_1(f_j,F_j)_{Q_2}\end{matrix}\right]$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}\, H^{M+m+1,\ N\ :\ M_1,N_1;\ M_2,N_2}_{P+m+1,\ Q+m+1:\ P_1,Q_1;\ P_2,Q_2}\left[\begin{matrix}x\\ y\end{matrix}\Big|\begin{matrix}{}_1(a_j;\alpha_j,A_j)_P\,,\ (v_1-1;\mu'_1,\mu''_1)\,,\,{}_1(\delta_i+r;\eta'_i,\eta''_i)_m:\ {}_1(c_j;C_j)_{P_1};\,{}_1(e_j,E_j)_{P_2}\\ (v_1;\mu'_1,\mu''_1),(v_1+r-1;\mu'_1,\mu''_1),\ {}_2(v_i+r;\mu'_i,\mu''_i)_m\,,\,{}_1(b_j;\beta_j,B_j)_Q:\ {}_1(d_j;D_j)_{Q_1};\,{}_1(f_j,F_j)_{Q_2}\end{matrix}\right]$$

$$+\sum_{r=0}^{\infty}\frac{t^{r+1}}{r!}$$

$$H^{M+m,\ N\ :\ M_1,N_1;\ M_2,N_2}_{P+m,\ Q+m\ :\ P_1,Q_1;\ P_2,Q_2}\left[\begin{matrix}x\\ y\end{matrix}\Big|\begin{matrix}{}_1(a_j;\alpha_j,A_j)_P\,,\,{}_1(\delta_i+r+1;\eta'_i,\eta''_i)_m:\ {}_1(c_j;C_j)_{P_1};\,{}_1(e_j,E_j)_{P_2}\\ (v_1+r;\mu'_1,\mu''_1),{}_2(v_i+r+1;\mu'_i,\mu''_i)_m\,,\,{}_1(b_j;\beta_j,B_j)_Q:\ {}_1(d_j;D_j)_{Q_1};\,{}_1(f_j,F_j)_{Q_2}\end{matrix}\right] \qquad (1.4.1)$$

Where m>0.

Proof : From the Mellin Barnes contour integral, change the order of the integration and summation and use the property of the Gamma function to obtain:

$$\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Psi(s_1,s_2)\theta(s_1)\,\emptyset(s_2)\,x^{s_1}y^{s_2}\,\frac{\prod_{i=1}^{m}\Gamma(v_i-\mu'_i s_1-\mu''_i s_2)}{\prod_{i=1}^{m}\Gamma(\delta_i-\eta'_i s_1-\eta''_i s_2)}$$

$$\left[\sum_{r=0}^{\infty}\frac{t^r}{r!}\,\frac{\prod_{i=1}^{m}(v_i-\mu'_i s_1-\mu''_i s_2)_r}{\prod_{i=1}^{m}(\delta_i-\eta'_i s_1-\eta''_i s_2)_r}\right]ds_1 ds_2 \qquad (1.4.2)$$

By the definition of hyper-geometric function the expression (1.4.2) becomes

$$\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Psi(s_1,s_2)\theta(s_1)\,\emptyset(s_2)\,x^{s_1}y^{s_2}\,\frac{\prod_{i=1}^{m}\Gamma(v_i-\mu'_i s_1-\mu''_i s_2)}{\prod_{i=1}^{m}\Gamma(\delta_i-\eta'_i s_1-\eta''_i s_2)}$$

$$_mF_m\left[\begin{matrix}{}_1(v_i-\mu'_i s_1-\mu''_i s_2)_m;\\ {}_1(\delta_i-\eta'_i s_1-\eta''_i s_2)_m;\end{matrix}\ t\right] \qquad (1.4.3)$$

Now for p=q=m on (1.4.3) expressing the resulting hyper-geometric functions as series , changing the order of integration and summation and then interpreting the two double integrals we arrive at the required result.

SPECIAL CASE

When m=2, the identity (1.4.1) reduces to the following form:

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}\, H^{M+2,\ N\ :\ M_1,N_1;\ M_2,N_2}_{P+2,\ Q+2\ :\ P_1,Q_1;\ P_2,Q_2}\left[\begin{matrix}x\\ y\end{matrix}\Big|\begin{matrix}{}_1(a_j;\alpha_j,A_j)_P,(\delta+r;\eta_1,\eta_2),\ (\sigma+r;\xi_1,\xi_2):\ {}_1(c_j;C_j)_{P_1};\,{}_1(e_j,E_j)_{P_2}\\ (v+r;\mu_1,\mu_2),(\rho+r;\gamma_1,\gamma_2),\ {}_1(b_j;\beta_j,B_j)_Q:\ {}_1(d_j;D_j)_{Q_1};\,{}_1(f_j,F_j)_{Q_2}\end{matrix}\right]$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}\, H^{M+3,\ N\ :\ M_1,N_1;\ M_2,N_2}_{P+3,\ Q+3:\ P_1,Q_1;\ P_2,Q_2}\left[\begin{matrix}x\\ y\end{matrix}\Big|\begin{matrix}{}_1(a_j;\alpha_j,A_j)_P,\ (v-1;\mu_1,\mu_2),(\delta+r;\eta_1,\eta_2),(\sigma+r;\xi_1,\xi_2):\ {}_1(c_j;C_j)_{P_1};\,{}_1(e_j,E_j)_{P_2}\\ (v;\mu_1,\mu_2),(v+r-1;\mu_1,\mu_2),(\rho+r;\gamma_1,\gamma_2),{}_1(b_j;\beta_j,B_j)_Q:\ {}_1(d_j;D_j)_{Q_1};\,{}_1(f_j,F_j)_{Q_2}\end{matrix}\right]$$

$$+\sum_{r=0}^{\infty}\frac{t^{r+1}}{r!}\, H^{M+2,\ N\ :\ M_1,N_1;\ M_2,N_2}_{P+2,\ Q+2\ :\ P_1,Q_1;\ P_2,Q_2}\left[\begin{matrix}x\\ y\end{matrix}\Big|\begin{matrix}{}_1(a_j;\alpha_j,A_j)_P,(\delta+r+1;\eta_1,\eta_2),(\sigma+r+1;\xi_1,\xi_2):\ {}_1(c_j;C_j)_{P_1};\,{}_1(e_j,E_j)_{P_2}\\ (v+r;\mu_1,\mu_2),(\rho+r+1;\gamma_1,\gamma_2),\ {}_1(b_j;\beta_j,B_j)_Q:\ {}_1(d_j;D_j)_{Q_1};\,{}_1(f_j,F_j)_{Q_2}\end{matrix}\right] \qquad (1.4.4)$$

Taking M=N=P=Q=0 ; $M_2=P_2$; $E_j = F_j = 1$; $f_1=0$; $\eta_i''= \mu''=0$ with $y\to 0$, and renaming the parameters (1.4.1) gives rise to:

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}\, H^{m+k,\ n}_{p+k,q+k}\left[x\Big|\begin{matrix}{}_1(a_j,\alpha_j)_p,\,{}_1(\delta_i+r,\ \eta_i)_k\\ {}_1(v_i+r,\ \mu_i)_k\,,\,{}_1(b_j,\beta_j)_q\end{matrix}\right]$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}\, H^{m+k+1,\ n}_{p+k+1,q+k+1}\left[x\Big|\begin{matrix}{}_1(a_j,\alpha_j)_p,(v_1-1,\ \mu_1),\ {}_1(\delta_i+r,\ \eta_i)_k\\ (v_1,\ \mu_1),\ (v_1+r-1,\ \mu_1),\ {}_2(v_i+r,\ \mu_i)_k\,,\,{}_1(b_j,\beta_j)_q\end{matrix}\right]$$

$$+\sum_{r=0}^{\infty}\frac{t^{r+1}}{r!}\, H^{m+k,\ n}_{p+k,q+k}\left[x\Big|\begin{matrix}{}_1(a_j,\alpha_j)_p,\,{}_1(\delta_i+r+1,\ \eta_i)_k\\ (v_1+r,\ \mu_1),{}_2(v_i+r+1,\ \mu_i)_k,{}_1(b_j,\beta_j)_q\end{matrix}\right],\ k>0 \qquad (1.4.5)$$

When k=1, (1.4.5) becomes:

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}\, H^{m+1,\ n}_{p+1,q+1}\left[x\Big|\begin{matrix}{}_1(a_j,\alpha_j)_p,(\delta+r,\ \eta)\\ (v+r,\ \mu),{}_1(b_j,\beta_j)_q\end{matrix}\right]$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}\, H^{m+2,\ n}_{p+2,q+2}\left[x\Big|\begin{matrix}{}_1(a_j,\alpha_j)_p,(v-1,\ \mu),\ (\delta+r,\ \eta)\\ (v,\ \mu),\ (v+r-1,\ \mu),{}_1(b_j,\beta_j)_q\end{matrix}\right]+$$

$$\sum_{r=0}^{\infty}\frac{t^{r+1}}{r!}\, H^{m+1,\ n}_{p+1,q+1}\left[x\Big|\begin{matrix}{}_1(a_j,\alpha_j)_p,(\delta+r+1,\ \eta)\\ (v+r,\ \mu),{}_1(b_j,\beta_j)_q\end{matrix}\right] \qquad (1.4.6)$$

IDENTITY-II

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}H$$

$$^{M+m,\ N+m\ :\ M_1,N_1;\ M_2,N_2}_{P+2m,\ Q+m\ :\ P_1,Q_1;\ P_2,Q_2}\left[\begin{matrix}x\\ y\end{matrix}\Big|\begin{matrix}{}_1(\delta_i-r;\eta'_i,\eta''_i)_m,{}_1(a_j;\alpha_j,A_j)_P\,,\,{}_1(\varsigma_i+r;\lambda'_i,\lambda''_i)_m,{}_1(c_j;C_j)_{P_1};\,{}_1(e_j,E_j)_{P_2}\\ {}_1(v_i+r;\mu'_i,\mu''_i)_m\,,\,{}_1(b_j;\beta_j,B_j)_Q:\ {}_1(d_j;D_j)_{Q_1};\,{}_1(f_j,F_j)_{Q_2}\end{matrix}\right]$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}H$$

$$^{M+m,\ N+m+1\ :\ M_1,N_1;\ M_2,N_2}_{P+2m+1,\ Q+m+1:\ P_1,Q_1;\ P_2,Q_2}\left[\begin{matrix}x\\ y\end{matrix}\Big|\begin{matrix}(\delta_1;\eta'_1,\eta''_1),(\delta_1-r+1;\eta'_1,\eta''_1),{}_2(\delta_i-r;\eta'_i,\eta''_i)_m,{}_1(a_j;\alpha_j,A_j)_P\,,\,{}_1(\varsigma_i+r;\lambda'_i,\lambda''_i)_m:\ {}_1(c_j;C_j)_{P_1};\,{}_1(e_j,E_j)_{P_2}\\ {}_1(v_i+r;\mu'_i,\mu''_i)_m,(b_j;\beta_j,B_j)_Q,\ (\delta_1+1;\eta'_1,\eta''_1):\ {}_1(d_j;D_j)_{Q_1};\,{}_1(f_j,F_j)_{Q_2}\end{matrix}\right]$$

$$+\sum_{r=0}^{\infty}\frac{t^{r+1}}{r!}H$$

$$^{M+m,\ N\ :\ M_1,N_1;\ M_2,N_2}_{P+2m,\ Q+m\ :\ P_1,Q_1;\ P_2,Q_2}\left[\begin{matrix}x\\ y\end{matrix}\Big|\begin{matrix}(\delta_1-r;\eta'_1,\eta''_1),{}_2(\delta_i-r-1;\eta'_i,\eta''_i)_m,{}_1(a_j;\alpha_j,A_j)_P\,,\,{}_1(\varsigma_i+r+1;\lambda'_i,\lambda''_i)_m:\ {}_1(c_j;C_j)_{P_1};\,{}_1(e_j,E_j)_{P_2}\\ {}_1(v_i+r+1;\mu'_i,\mu''_i)_m\,,\,{}_1(b_j;\beta_j,B_j)_Q:\ {}_1(d_j;D_j)_{Q_1};\,{}_1(f_j,F_j)_{Q_2}\end{matrix}\right];\ m>0$$

$$(1.4.7)$$

Proof is obtained in similar lines to (1.4.1), on using (1.4.6) for p=2m and q=m.

SPECIAL CASE

Taking M=N=P=Q=0 ; $M_2=1$; $N_2=P_2$; $E_j = F_j = 1$; $f_1=0$; $\eta_i''=\mu_i''=\lambda_i''=0$ with $y\to 0$, and renaming the parameters (1.4.7) gives rise to:

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}\, H^{m+k,\ n+k}_{P+2k,\ q+k}\left[x\Big|\begin{matrix}{}_1(\delta_i-r;\eta_i)_k,{}_1(a_j;\alpha_j)_P\,,\,{}_1(\varsigma_i+r,\lambda_i)_k,\\ {}_1(v_i+r;\mu_i)_k\,,\,{}_1(b_j;\beta_j)_q\end{matrix}\right]$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}\, H^{m+k,\ n+k+1}_{p+2k+1,\ q+k+1}\left[x\Big|\begin{matrix}(\delta_1;\eta_1),(\delta_1-r+1;\eta_1),{}_2(\delta_i-r;\eta_i)_k,{}_1(a_j;\alpha_j)_P\,,\,{}_1(\varsigma_i+r,\lambda_i)_k\\ {}_1(v_i+r;\mu_i)_k\,,\,{}_1(b_j;\beta_j)_q,(\delta_1+1;\eta_1)\end{matrix}\right]$$

$$+\sum_{r=0}^{\infty}\frac{t^{r+1}}{r!}\, H^{m+k,n+k}_{p+2k,q+k}\left[x\Big|\begin{matrix}(\delta_1-r;\eta_1),{}_2(\delta_i-r-1;\eta_i)_k,{}_1(a_j;\alpha_j)_P\,,\,{}_1(\varsigma_i+r+1,\ \lambda_i)_k\\ {}_1(v_i+r+1;\mu_i)_k\,,\,{}_1(b_j;\beta_j)_q\end{matrix}\right];\ \text{where } k>0$$

(1.4.8

When k=2 ,(1.4.8) becomes

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}\, H^{m+2,\ n+2}_{P+4,\ q+2}\left[x\Big|\begin{matrix}(\delta-r;\eta),(\sigma-r,\ \xi),{}_1(a_j;\alpha_j)_P\,,\,{}_1(\varsigma+r,\lambda),(\tau+r,\ \zeta)\\ (v+r,\ \mu),\ (\rho+r,\ \gamma),\ {}_1(b_j;\beta_j)_q\end{matrix}\right]$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}\, H^{m+2,\ n+3}_{p+5,\ q+3}\left[x\Big|\begin{matrix}(\delta,\ \eta),\ (\delta-r+1;\eta),\ (\sigma-r,\ \xi),{}_1(a_j;\alpha_j)_P,\ (\varsigma+r,\ \lambda),\ (\tau+r,\ \zeta\\ (v+r,\ \mu),\ (\rho+r,\ \gamma),\ {}_1(b_j;\beta_j)_q,(\delta+1,\ \eta),\end{matrix}\right]$$

$$+\sum_{r=0}^{\infty}\frac{t^{r+1}}{r!}\, H^{m+2,n+2}_{p+4,q+2}\left[x\Big|\begin{matrix}(\delta-r,\ \eta),\ (\delta-r;\eta),\ (\sigma-r-1,\ \xi),{}_1(a_j;\alpha_j)_P,\ (\varsigma+r+1,\ \lambda),\ (\tau+r+1,\ \zeta\\ (v+r+1,\ \mu),\ (\rho+r+1,\ \gamma),\ {}_1(b_j;\beta_j)_q,\end{matrix}\right];\qquad (1.4.9)$$

IDENTITY-III

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}H$$

$$^{M+m,\ N+m\ :\ M_1,N_1;\ M_2,N_2}_{P+m,\ Q+2m:\ P_1,Q_1;\ P_2,Q_2}\left[\begin{matrix}x\\ y\end{matrix}\Big|\begin{matrix}{}_1(\delta_i-r;\eta'_i,\eta''_i)_m,{}_1(a_j;\alpha_j,A_j)_P\,,\,{}_1(c_j;C_j)_{P_1};\,{}_1(e_j,E_j)_{P_2}\\ {}_1(v_i+r;\mu'_i,\mu''_i)_m\,,\,{}_1(b_j;\beta_j,B_j)_Q\ ,\,{}_1(\varsigma_i-r;\lambda'_i,\lambda''_i)_m:\ {}_1(d_j;D_j)_{Q_1};\,{}_1(f_j,F_j)_{Q_2}\end{matrix}\right]$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}H^{M+m,\ N+m+1\ :\ M_1,N_1;M_2,N_2}_{P+m+1,\ Q+2m+1:P_1,Q_1;P_2,Q_2}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(\delta_1;\eta'_1,\eta''_1),(\delta_1-r+1;\eta'_1,\eta''_1),{}_2(\delta_i-r;\eta'_i,\eta''_i)_m,{}_1(a_j;\alpha_j,A_j)_P:{}_1(c_j;C_j)_{P1};{}_1(e_j,E_j)_{P2}\\{}_1(v_i+r;\mu'_i,\mu''_i)_m,(b_j;\beta_j,B_j)_Q,(\delta_1+1;\eta'_1,\eta''_1),{}_1(\varsigma_i+r;\lambda'_i,\lambda''_i)_m:{}_1(d_j;D_j)_{Q1};{}_1(f_j,F_j)_{Q2}\end{matrix}\right]$$

$$+\sum_{r=0}^{\infty}\frac{t^{r+1}}{r!}H^{M+m,\ N+m\ :\ M_1,N_1;M_2,N_2}_{P+m,\ Q+2m:P_1,Q_1;P_2,Q_2}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(\delta_1-r;\eta'_1,\eta''_1),{}_2(\delta_i-r-1;\eta'_i,\eta''_i)_m,{}_1(a_j;\alpha_j,A_j)_P:{}_1(c_j;C_j)_{P1};{}_1(e_j,E_j)_{P2}\\{}_1(v_i+r+1;\mu'_i,\mu''_i)_m,{}_1(b_j;\beta_j,B_j)_Q,{}_1(\varsigma_i+r;\lambda'_i,\lambda''_i)_m:{}_1(d_j;D_j)_{Q1};{}_1(f_j,F_j)_{Q2}\end{matrix}\right];\ m>0 \tag{1.4.10}$$

Proceeding similar lines to (1.4.1), the proof of (1.4.10) is obtained on using (1.4.6) for p=2m and q=m.

SPECIAL CASE

Taking M=N=P=Q=0 ; M_2=1 ;N_2=P_2 ; E_j = F_j = 1 ; f_1=0 ; $\eta_i''=\mu_i''=\lambda_i''$=0with y → 0, and renaming the parameters (1.4.7) gives rise to:

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}H^{m+k,\ n+k}_{P+k,\ q+2k}\left[x\middle|\begin{matrix}{}_1(\delta_i-r;\eta_i)_k,{}_1(a_j;\alpha_j)_P,\\{}_1(v_i+r;\mu_i)_k,{}_1(b_j;\beta_j)_q,{}_1(\varsigma_i-r,\ \lambda_i)_k\end{matrix}\right]$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}H^{m+k,\ n+k+1}_{p+k+1,\ q+2k+1}\left[x\middle|\begin{matrix}(\delta_1,\ \eta_1),(\delta_1-r+1;\eta_1),{}_2(\delta_i-r;\eta_i)_k,{}_1(a_j;\alpha_j)_P\\{}_1(v_i+r;\mu_i)_k,{}_1(b_j;\beta_j)_q,(\delta_1+1;\eta_1),{}_1(\varsigma_i-r,\lambda_i)_k\end{matrix}\right]$$

$$+\sum_{r=0}^{\infty}\frac{t^{r+1}}{r!}H^{m+k,n+k}_{p+k,q+2k}\left[x\middle|\begin{matrix}(\delta_1-r;\eta_1),{}_2(\delta_i-r-1;\eta_i)_k,{}_1(a_j;\alpha_j)_P\\{}_1(v_i+r+1;\mu_i)_k,{}_1(b_j;\beta_j)_q,\ {}_1(\varsigma_i-r-1,\ \lambda_i)_k\end{matrix}\right];$$

where k>0 (1.4.11)

On taking k=2, (1.4.11) reduces to the following identity:

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}H^{m+2,\ n+2}_{P+2,\ q+4}\left[x\middle|\begin{matrix}(\delta-r;\eta),(\sigma-r,\ \xi),{}_1(a_j;\alpha_j)_P\\(v+r,\ \mu),\ (\rho+r,\ \gamma),\ {}_1(b_j;\beta_j)_q,(\varsigma-r,\lambda),(\tau-r,\ \zeta)\end{matrix}\right]$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}H^{m+2,\ n+3}_{p+3,\ q+3}\left[x\middle|\begin{matrix}(\delta,\ \eta),\ (\delta-r+1;\eta),\ (\sigma-r,\ \xi),{}_1(a_j;\alpha_j)_P\\(v+r,\ \mu),\ (\rho+r,\ \gamma),\ {}_1(b_j;\beta_j)_q,(\delta+1,\ \eta),\ (\varsigma-r,\ \lambda),\ (\tau-r,\ \zeta)\end{matrix}\right]$$

$$+\sum_{r=0}^{\infty}\frac{t^{r+1}}{r!}H^{m+2,n+2}_{p+2,q+4}\left[x\middle|\begin{matrix}(\delta-r;\eta),\ (\sigma-r-1,\ \xi),{}_1(a_j;\alpha_j)_P,\\(v+r+1,\ \mu),\ (\rho+r+1,\ \gamma),\ {}_1(b_j;\beta_j)_q,\ ,\ (\varsigma-r-1,\ \lambda),\ (\tau-r-1,\ \zeta)\end{matrix}\right]; \tag{1.4.12}$$

When k=1, (1.4.11) gives rise to the identity:

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}H^{m+1,\ n+1}_{P+1,\ q+2}\left[x\middle|\begin{matrix}(\delta-r;\eta),\ {}_1(a_j;\alpha_j)_P\\(v+r,\ \mu),\ {}_1(b_j;\beta_j)_q,(\varsigma-r,\lambda)\end{matrix}\right]$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}H^{m+1,\ n+2}_{p+2,\ q+3}\left[x\middle|\begin{matrix}(\delta,\ \eta),\ (\delta-r+1;\eta),{}_1(a_j;\alpha_j)_P\\(v+r,\ \mu),\ {}_1(b_j;\beta_j)_q,(\delta+1,\ \eta),\ (\varsigma-r,\ \lambda),\end{matrix}\right]$$

$$+\sum_{r=0}^{\infty}\frac{t^{r+1}}{r!}H^{m+1,n+1}_{p+1,q+2}\left[x\middle|\begin{matrix}(\delta-r;\eta),{}_1(a_j;\alpha_j)_P,\\(v+r+1,\ \mu),\ {}_1(b_j;\beta_j)_q,\ ,\ (\varsigma-r-1,\ \lambda)\end{matrix}\right]; \tag{1.4.13}$$

IDENTITY-IV

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}H^{M+m,\ N\ :\ M_1,N_1;M_2,N_2}_{P+m,\ Q+m:P_1,Q_1;P_2,Q_2}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}{}_1(a_j;\alpha_j,A_j)_P,{}_1(\delta_i+r;\eta'_i,\eta''_i)_m:{}_1(c_j;C_j)_{P1};{}_1(e_j,E_j)_{P2}\\{}_1(v_i+r;\mu'_i,\mu''_i)_m,{}_1(b_j;\beta_j,B_j)_Q:{}_1(d_j;D_j)_{Q1};{}_1(f_j,F_j)_{Q2}\end{matrix}\right]$$

=

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}H^{M+m+1,\ N\ :\ M_1,N_1;M_2,N_2}_{P+m+1,\ Q+m+1:P_1,Q_1;P_2,Q_2}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}{}_1(a_j;\alpha_j,A_j)_P,\ (v_1-V_m+1;\mu'_1-\mu'_m,\mu''_1-\mu''_m),{}_1(\delta_i+r;\eta'_i,\eta''_i)_m:{}_1(c_j;C_j)_{P1};{}_1(e_j,E_j)_{P2}\\(v_1+r+1;\mu'_1,\mu''_1),{}_2(v_i+r;\mu'_i,\mu''_i)_m,(v_1-V_m+1;\mu'_1-\mu'_m,\mu''_1-\mu''_m),{}_1(b_j;\beta_j,B_j)_Q:{}_1(d_j;D_j)_{Q1};{}_1(f_j,F_j)_{Q2}\end{matrix}\right]$$

$$-\sum_{r=0}^{\infty}\frac{t^r}{r!}H^{M+m+1,\ N\ :\ M_1,N_1;M_2,N_2}_{P+m+1,\ Q+m+1:P_1,Q_1;P_2,Q_2}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}{}_1(a_j;\alpha_j,A_j)_P,(v_1-V_m+1;\mu'_1-\mu'_m,\mu''_1-\mu''_m),{}_1(\delta_i+r+1;\eta'_i,\eta''_i)_m:{}_1(c_j;C_j)_{P1};{}_1(e_j,E_j)_{P2}\\(v_i+r;\mu'_1,\mu''_1)_{m-1},(v_m+r+1;\mu'_m,\mu''_m),(v_1-V_m+1;\mu'_1-\mu'_m,\mu''_1-\mu''_m),{}_1(b_j;\beta_j,B_j)_Q:{}_1(d_j;D_j)_{Q1};{}_1(f_j,F_j)_{Q2}\end{matrix}\right] \tag{1.4.14}$$

Where m≥2

The proof is obtained in similar lines to (1.4.1) p=q=m.

SPECIAL CASE

When m=2, the identity (1.4.14) takes the following form:

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}H^{M+2,\ N\ :\ M_1,N_1;M_2,N_2}_{P+2,\ Q+2:P_1,Q_1;P_2,Q_2}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}{}_1(a_j;\alpha_j,A_j)_P,(\delta+r;\eta_1,\eta_2),\ (\sigma+r;\xi_1,\xi_2):{}_1(c_j;C_j)_{P1};{}_1(e_j,E_j)_{P_2}\\(v+r;\mu_1,\mu_2),(\rho+r;\gamma_1,\gamma_2),\ {}_1(b_j;\beta_j,B_j)_Q:{}_1(d_j;D_j)_{Q1};{}_1(f_j,F_j)_{Q_2}\end{matrix}\right]$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}H^{M+3,\ N\ :\ M_1,N_1;M_2,N_2}_{P+3,\ Q+3:P_1,Q_1;P_2,Q_2}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}{}_1(a_j;\alpha_j,A_j)_P,\ (v-\rho+1;\mu_1-\gamma_1,\mu_2-\gamma_2),(\delta+r;\eta_1,\eta_2),(\sigma+r;\xi_1,\xi_2):{}_1(c_j;C_j)_{P1};{}_1(e_j,E_j)_{P_2}\\(v+r+1;\mu_1,\mu_2),(\rho+r;\gamma_1,\gamma_2),(v-\rho;\mu_1-\gamma_1,\mu_2-\gamma_2),\ {}_1(b_j;\beta_j,B_j)_Q:{}_1(d_j;D_j)_{Q_1};{}_1(f_j,F_j)_{Q_2}\end{matrix}\right]$$

$$-\sum_{r=0}^{\infty}\frac{t^r}{r!}H^{M+3,\ N\ :\ M_1,N_1;M_2,N_2}_{P+3,\ Q+3:P_1,Q_1;P_2,Q_2}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}{}_1(a_j;\alpha_j,A_j)_P,(v-\rho+1;\mu_1-\gamma_1,\mu_2-\gamma_2),(\delta+r;\eta_1,\eta_2),(\sigma+r;\xi_1,\xi_2):{}_1(c_j;C_j)_{P1};{}_1(e_j,E_j)_{P2}\\(v+r;\mu_1,\mu_2),(\rho+r+1;\gamma_1,\gamma_2),(v-\rho;\mu_1-\gamma_1,\mu_2-\gamma_2),\ {}_1(b_j;\beta_j,B_j)_Q:{}_1(d_j;D_j)_{Q_1};{}_1(f_j,F_j)_{Q_2}\end{matrix}\right] \tag{1.4.15}$$

Taking M=N=P=Q=0 ; M_2=1; N_2 ; P_2 ; E_j = F_j = 1 ; f_1=0 ; $\eta_i''=\mu_i''$=0 with y→ 0, and renaming the parameters (1.4.14) gives rise to:

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}H^{m+k,\ n}_{p+k,q+k}\left[x\middle|\begin{matrix}{}_1(a_j,\alpha_j)_p,{}_1(\delta_i+r,\ \eta_i)_k\\{}_1(v_i+r,\ \mu_i)_k,{}_1(b_j,\beta_j)_q\end{matrix}\right]$$

$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}H^{m+k+1,\ n}_{p+k+1,q+k+1}\left[x\middle|\begin{matrix}{}_1(a_j,\alpha_j)_p,(v_1-v_k+1,\ \mu_1-\mu_k),\ {}_1(\delta_i+r,\ \eta_i)_k\\(v_1+r+1,\ \mu_1),\ {}_2(v_i+r,\ \mu_i)_k,\ (v_1-v_k,\ \mu_1-\mu_k),{}_1(b_j,\beta_j)_q\end{matrix}\right]$$

$$-\sum_{r=0}^{\infty}\frac{t^r}{r!}H^{m+k+1,\ n}_{p+k+1,q+k+1}\left[x\middle|\begin{matrix}{}_1(a_j,\alpha_j)_p,(v_1-v_k+1,\ \mu_1-\mu_k),\ {}_1(\delta_i+r,\ \eta_i)_k\\(v_1+r,\ \mu_i)_{k-1},(v_k+r+1,\ \mu_k),(v_1-v_k,\ \mu_1-\mu_k),\ {}_1(b_j,\beta_j)_q\end{matrix}\right]$$

, k≥2 (1.4.16)

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