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Mathematics

ARITHMETIC OPERATIONS ON POLYGONAL FUZZY NUMBERS

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ABSTRACT In decision-making process, one has to order certain attributes based on their relative importance. In many such situations, there is need of help of experts who have knowledge in that particular area. Usually experts' opinion is obtained as linguistic variables which can be fuzzified using some fuzzy scaling. In this paper, fuzzy numbers with different number of vertices called polygonal fuzzy numbers were considered and included basic arithmetic operations of polygonal fuzzy numbers by means of α-cuts. The formulas derived for finding membership function associated with any kind of fuzzy number and corresponding alpha cut is very useful in using fuzzy logic in decision-making which help us to compile the experts' judgements while developing decision-making models. A membership function for a fuzzy number of the form $(a_1, a_2, a_3, \ldots, a_n)$ was developed which helps to define membership function for any value of n. Different formulas had been developed for odd and even values of n.

KEYWORDS : Polygonal fuzzy numbers, membership functions, α-cuts, Arithmetic operations

1. INTRODUCTION

To represent fuzzy sets, its membership functions are to be defined. A polygonal fuzzy number and its operations are very useful in Fuzzy Multi-Criteria Decision-Making Process. To express human thinking more clearly, we can use fuzzy numbers with more vertices and it provides more flexibility to handle ambiguity. When we express linguistic terms as fuzzy numbers while developing Fuzzy Multi-Criteria Decision-Making models, we can use any type of fuzzy number as the case may be. Fuzzy numbers with more vertices provide higher flexibility to the decision maker to express his own linguistics which ensure better handling of the Fuzzy Multi-Criteria Decision-Making Problem.

2. PRELIMINARIES

Definition 2.1 (Klir and Yuan, 2001)

A fuzzy set A in a universe of discourse X is defined as the set of pairs, $A = \{(x, \mu_A(x)) : x \in X\}$, where $\mu_A(x) : X \to [0,1]$ is called the membership value of $x \in X$ in the fuzzy set A.

Definition 2.2 (Klir and Yuan, 2001)

The α-cut, $\alpha \in (0,1]$ of a fuzzy number A is a crisp set defined as $A(\alpha) = \{x \epsilon R : A(x) \geq \alpha\}$. Every A_α is a closed interval of the form $[A_L(\alpha), A_U(\alpha)]$.

Definition 2.3 (Ban and Lucian, 2014)

A fuzzy number A is a fuzzy subset of the real line; $A: R \to [0,1]$ satisfying the following properties:

(i) A is normal (i.e. there exists $x_0 \in R$ such that $A(x_0) = 1$);

(ii) A is fuzzy convex ;

(iii) A is upper semi continuous on R. *ie*; $\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that $A(x) - A(x_0) < \varepsilon$ whenever $|x - x_0| < \delta$;

(iv) The closure , $cl(supp(A))$ is compact.

Definition 2.4

Let A_1 and A_2 are two fuzzy numbers and an arithmetic operation $\circ$ is defined, where $\circ \in \{+, -, *, /\}$, in terms of α-cuts as follow

$$(A_1 \circ A_2)(\alpha) = A_1(\alpha) \circ A_2(\alpha)$$

$$= \{x \circ y \,/\, x \in A_1(\alpha), x \in A_2(\alpha)\};\ \alpha \in [0,1]$$

The result of an arithmetic operation $(A_1 \circ A_2) = \bigcup_{\alpha \in [0,1]} (A_1 \circ A_2)(\alpha)$ is always a fuzzy number (Skalna et al., 2015)

For any two fuzzy numbers A and B, addition and subtraction operations result again in fuzzy numbers.

3. POLYGONAL FUZZY NUMBERS

Definition 3.1: Polygonal Fuzzy Number with n vertices, n is an odd number

A polygonal fuzzy number with n vertices (n is odd) is a fuzzy number of the form $P = (a_1, a_2, a_3, \ldots, a_n)$ with membership function

$$\mu_P(x) = \begin{cases}
0 & \text{if } x < a_1 \\
P_1(x) = \frac{2}{n-1}\left(\frac{x-a_1}{a_2-a_1}\right) & \text{if } a_1 \leq x \leq a_2 \\
P_2(x) = \frac{2}{n-1} + \frac{2}{n-1}\left(\frac{x-a_2}{a_3-a_2}\right) & \text{if } a_2 \leq x \leq a_3 \\
P_3(x) = \frac{4}{n-1} + \frac{2}{n-1}\left(\frac{x-a_3}{a_4-a_3}\right) & \text{if } a_3 \leq x \leq a_4 \\
P_4(x) = \frac{6}{n-1} + \frac{2}{n-1}\left(\frac{x-a_4}{a_5-a_4}\right) & \text{if } a_4 \leq x \leq a_5 \\
\ldots\ldots\ldots\ldots\ldots\ldots & \\
\ldots\ldots\ldots\ldots\ldots\ldots & \\
\ldots\ldots\ldots\ldots\ldots\ldots & \\
P_{\frac{n-1}{2}}(x) = \frac{n-3}{n-1} + \frac{2}{n-1}\left(\frac{x-a_{\frac{n-1}{2}}}{a_{\frac{n+1}{2}}-a_{\frac{n-1}{2}}}\right) & \text{if } a_{\frac{n-1}{2}} \leq x \leq a_{\frac{n+1}{2}} \\
P_{\frac{n-1}{2}}'(x) = 1 - \frac{2}{n-1}\left(\frac{x-a_{\frac{n+1}{2}}}{a_{\frac{n+3}{2}}-a_{\frac{n+1}{2}}}\right) & \text{if } a_{\frac{n+1}{2}} \leq x \leq a_{\frac{n+3}{2}} \\
P_{\frac{n-3}{2}}'(x) = \frac{n-3}{n-1} - \frac{2}{n-1}\left(\frac{x-a_{\frac{n+3}{2}}}{a_{\frac{n+5}{2}}-a_{\frac{n+3}{2}}}\right) & \text{if } a_{\frac{n+3}{2}} \leq x \leq a_{\frac{n+5}{2}} \\
P_{\frac{n-5}{2}}'(x) = \frac{n-5}{n-1} - \frac{2}{n-1}\left(\frac{x-a_{\frac{n+5}{2}}}{a_{\frac{n+7}{2}}-a_{\frac{n+5}{2}}}\right) & \text{if } a_{\frac{n+5}{2}} \leq x \leq a_{\frac{n+7}{2}} \\
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\ldots\ldots\ldots\ldots\ldots\ldots & \\
P_4'(x) = \frac{8}{n-1} - \frac{2}{n-1}\left(\frac{x-a_{n-4}}{a_{n-3}-a_{n-4}}\right) & \text{if } a_{n-4} \leq x \leq a_{n-3} \\
P_3'(x) = \frac{6}{n-1} - \frac{2}{n-1}\left(\frac{x-a_{n-3}}{a_{n-2}-a_{n-3}}\right) & \text{if } a_{n-3} \leq x \leq a_{n-2} \\
P_2'(x) = \frac{4}{n-1} - \frac{2}{n-1}\left(\frac{x-a_{n-2}}{a_{n-1}-a_{n-2}}\right) & \text{if } a_{n-2} \leq x \leq a_{n-1} \\
P_1'(x) = \frac{2}{n-1}\left(\frac{a_n-x}{a_n-a_{n-1}}\right) & \text{if } a_{n-1} \leq x \leq a_n \\
0 & \text{if } x > a_n
\end{cases}$$

Definition 3.2

The α-cut of the polygonal fuzzy number $P = (a_1, a_2, a_3, \dots, a_n)$ (n is an odd number) of the universe of discourse X is defined as $P(\alpha) = \{x \in X / \mu_P(x) \geq \alpha\}$ where $\alpha \in [0,1]$ and $P_1(x), P_2(x), \dots, P_{\frac{n-1}{2}}(x)$ are bounded and continuous increasing functions over $\left[0, \frac{2}{n-1}\right], \left[\frac{2}{n-1}, \frac{4}{n-1}\right], \left[\frac{4}{n-1}, \frac{6}{n-1}\right], \dots, \left[\frac{n-3}{n-1}, 1\right]$ respectively and $P_1'(x),\ P_2'(x), \dots,\ P_{\frac{n-1}{2}}'(x)$ are bounded and continuous decreasing functions over $\left[0, \frac{2}{n-1}\right],\ \left[\frac{2}{n-1}, \frac{4}{n-1}\right],\ \left[\frac{4}{n-1}, \frac{6}{n-1}\right], \dots,\ \left[\frac{n-3}{n-1}, 1\right]$ respectively. Then,

$$P(\alpha) = \begin{cases} [P_1(\alpha),\ P_1'(\alpha)\ ]; & \alpha \in \left[0, \frac{2}{n-1}\right) \\ [P_2(\alpha),\ P_2'(\alpha)\ ]; & \alpha \in \left[\frac{2}{n-1}, \frac{4}{n-1}\right) \\ [P_3(\alpha),\ P_3'(\alpha)\ ]; & \alpha \in \left[\frac{4}{n-1}, \frac{6}{n-1}\right) \\ \dots\dots\dots & \\ \dots\dots\dots & \\ \dots\dots\dots & \\ \left[P_{\frac{n-1}{2}}(\alpha),\ P_{\frac{n-1}{2}}'(\alpha)\right]; & \alpha \in \left[\frac{n-3}{n-1}, 1\right) \end{cases}$$

Then $P_1(x) = \alpha$ and $P_1'(x) = \alpha$ gives the α-cut interval for $\alpha \in \left[0, \frac{2}{n-1}\right)$ as $\left[\frac{(n-1)}{2}\ (a_2 - a_1)\alpha + a_1,\ a_n - \frac{(n-1)}{2}\ (a_n - a_{n-1})\alpha\right]$

$P_2(x) = \alpha$ and $P_2'(x) = \alpha$ gives the α-cut interval for $\alpha \in \left[\frac{2}{n-1}, \frac{4}{n-1}\right)$ as

$$\left[\left(\frac{(n-1)\alpha - 2}{2}\right)(a_3 - a_2) + a_2,\quad \left(\frac{4 - (n-1)\alpha}{2}\right)(a_{n-1} - a_{n-2}) + a_{n-2}\ \right]$$

$P_3(x) = \alpha$ and $P_3'(x) = \alpha$ gives the α-cut interval for $\alpha \in \left[\frac{4}{n-1}, \frac{6}{n-1}\right)$ as

$$\left[\left(\frac{(n-1)\alpha-4}{2}\right)(a_4 - a_3) + a_3,\ \left(\frac{6-(n-1)\alpha}{2}\right)(a_{n-2} - a_{n-3}) + a_{n-3}\ \right]$$

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$P_{\frac{n-1}{2}}(x) = \alpha$ and $P_{\frac{n-1}{2}}'(x) = \alpha$ gives the α-cut interval for $\alpha \in \left[\frac{n-3}{n-1}, 1\right)$ as

$$\left[\left(\frac{(n-1)\alpha - (n-3)}{2}\right)\left(a_{\frac{n+1}{2}} - a_{\frac{n-1}{2}}\right) + a_{\frac{n-1}{2}},\quad \left(\frac{(1-\alpha)(n-1)}{2}\right)\left(a_{\frac{n+3}{2}} - a_{\frac{n+1}{2}}\right) + a_{\frac{n+1}{2}}\ \right]$$

$\therefore\ P(\alpha) =$

$$\begin{cases} \left[\frac{(n-1)}{2}\ (a_2 - a_1)\alpha + a_1,\ \ a_n - \frac{(n-1)}{2}\ (a_n - a_{n-1})\alpha\ \right]; & \alpha \in \left[0, \frac{2}{n-1}\right) \\ \left[\left(\frac{(n-1)\alpha-2}{2}\right)(a_3 - a_2) + a_2,\ \left(\frac{4-(n-1)\alpha}{2}\right)(a_{n-1} - a_{n-2}) + a_{n-2}\right]; & \alpha \in \left[\frac{2}{n-1}, \frac{4}{n-1}\right) \\ \left[\left(\frac{(n-1)\alpha-4}{2}\right)(a_4 - a_3) + a_3,\ \left(\frac{6-(n-1)\alpha}{2}\right)(a_{n-2} - a_{n-3}) + a_{n-3}\right]; & \alpha \in \left[\frac{4}{n-1}, \frac{6}{n-1}\right) \\ \dots\dots\dots & \\ \dots\dots\dots & \\ \dots\dots\dots & \\ \left[\left(\frac{(n-1)\alpha-(n-3)}{2}\right)\left(a_{\frac{n+1}{2}} - a_{\frac{n-1}{2}}\right) + a_{\frac{n-1}{2}},\ \left(\frac{(1-\alpha)(n-1)}{2}\right)\left(a_{\frac{n+3}{2}} - a_{\frac{n+1}{2}}\right) + a_{\frac{n+1}{2}}\right]; & \alpha \in \left[\frac{n-3}{n-1}, 1\right) \end{cases}$$

Example 3.1: Let $P = (a_1, a_2, a_3, \dots, a_{11})$ be a hendecagonal fuzzy number with membership function (Figure 2.1)

$$\mu_P(x) = \begin{cases} 0 & \text{if } x < a_1 \\ P_1(x) = \frac{1}{5}\left(\frac{x - a_1}{a_2 - a_1}\right) & \text{if } a_1 \leq x \leq a_2 \\ P_2(x) = \frac{1}{5} + \frac{1}{5}\left(\frac{x - a_2}{a_3 - a_2}\right) & \text{if } a_2 \leq x \leq a_3 \\ P_3(x) = \frac{2}{5} + \frac{1}{5}\left(\frac{x - a_3}{a_4 - a_3}\right) & \text{if } a_3 \leq x \leq a_4 \\ P_4(x) = \frac{3}{5} + \frac{1}{5}\left(\frac{x - a_4}{a_5 - a_4}\right) & \text{if } a_4 \leq x \leq a_5 \\ P_5(x) = \frac{4}{5} + \frac{1}{5}\left(\frac{x - a_5}{a_6 - a_5}\right) & \text{if } a_5 \leq x \leq a_6 \\ P_5'(x) = 1 - \frac{1}{5}\left(\frac{x - a_6}{a_7 - a_6}\right) & \text{if } a_6 \leq x \leq a_7 \\ P_4'(x) = \frac{4}{5} - \frac{1}{5}\left(\frac{x - a_7}{a_8 - a_7}\right) & \text{if } a_7 \leq x \leq a_8 \\ P_3'(x) = \frac{3}{5} - \frac{1}{5}\left(\frac{x - a_8}{a_9 - a_8}\right) & \text{if } a_8 \leq x \leq a_9 \\ P_2'(x) = \frac{2}{5} - \frac{1}{5}\left(\frac{x - a_9}{a_{10} - a_9}\right) & \text{if } a_9 \leq x \leq a_{10} \\ P_1'(x) = \frac{1}{5}\left(\frac{a_{11} - x}{a_{11} - a_{10}}\right) & \text{if } a_{10} \leq x \leq a_{11} \end{cases}$$

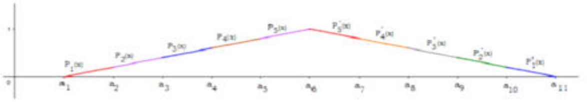

Figure 2.1

Hendecagonal fuzzy number

$P_1(x), P_2(x), P_3(x), P_4(x)$ and $P_5(x)$ are bounded and continuous increasing functions over [0,0.2], [0.2,0.4], [0.4,0.6], [0.6,0.8] and [0.8,1.0] respectively. $P_1'(x),\ P_2'(x),\ P_3'(x),\ P_4'(x)$ and $P_5'(x)$ are bounded and continuous decreasing functions over [0,0.2], [0.2,0.4], [0.4,0.6], [0.6,0.8] and [0.8,1.0] respectively.

Then its α-cut $P(\alpha)$ is given by

$P(\alpha) =$

$$\begin{cases} [a_1 + 5\alpha(a_2 - a_1),\ \ a_{11} - 5\alpha(a_{11} - a_{10})]; & \alpha \in [0,0.2) \\ [a_2 + (5\alpha - 1)(a_3 - a_2), a_9 + (2 - 5\alpha)(a_{10} - a_9)]; & \alpha \in [0.2,0.4) \\ [a_3 + (5\alpha - 2)(a_4 - a_3), a_8 + (3 - 5\alpha)(a_9 - a_8)]; & \alpha \in [0.4,0.6) \\ [a_4 + (5\alpha - 3)(a_5 - a_4), a_7 + (4 - 5\alpha)(a_8 - a_7)]; & \alpha \in [0.6,0.8) \\ [a_5 + (5\alpha - 4)(a_6 - a_5), a_6 + (5 - 5\alpha)(a_7 - a_6)]; & \alpha \in [0.8,1) \end{cases}$$

This shows that the α-cut of a polygonal fuzzy number with n (n is odd) vertices can be directly find out using the developed formula $P(\alpha)$ which helps us very much when we use fuzzy numbers.

Definition 3.3: Polygonal Fuzzy Number with n vertices, n is an even number

A polygonal fuzzy number with n (n is even) vertices of the universe of discourse X is a fuzzy number of the form $Q = (a_1, a_2, a_3, \dots, a_n)$ with membership function

$$\mu_Q(x) =$$

$$\begin{cases} 0 & \text{if } \quad x < a_1 \\ Q_1(x) = \frac{2}{n-2}\left(\frac{x - a_1}{a_2 - a_1}\right) & \text{if } a_1 \leq x \leq a_2 \end{cases}$$

$$
\begin{cases}
Q_2(x) = \frac{2}{n-2} + \frac{2}{n-2}\left(\frac{x-a_2}{a_3-a_2}\right) & \text{if } a_2 \le x \le a_3 \\
Q_3(x) = \frac{4}{n-2} + \frac{2}{n-2}\left(\frac{x-a_3}{a_4-a_3}\right) & \text{if } a_3 \le x \le a_4 \\
Q_4(x) = \frac{6}{n-2} + \frac{2}{n-2}\left(\frac{x-a_4}{a_5-a_4}\right) & \text{if } a_4 \le x \le a_5 \\
\quad \ldots\ldots\ldots\ldots\ldots\ldots & \\
\quad \ldots\ldots\ldots\ldots\ldots\ldots & \\
\quad \ldots\ldots\ldots\ldots\ldots\ldots & \\
Q_{\frac{n-2}{2}}(x) = \frac{n-4}{n-2} + \frac{2}{n-2}\left(\frac{x-a_{\frac{n-2}{2}}}{a_{\frac{n}{2}}-a_{\frac{n-2}{2}}}\right) & \text{if } a_{\frac{n-2}{2}} \le x \le a_{\frac{n}{2}} \\
1 & \text{if } a_{\frac{n}{2}} \le x \le a_{\frac{n+2}{2}} \\
Q_{\frac{n-2}{2}}'(x) = 1 - \frac{2}{n-2}\left(\frac{x-a_{\frac{n+2}{2}}}{a_{\frac{n+4}{2}}-a_{\frac{n+2}{2}}}\right) & \text{if } a_{\frac{n+2}{2}} \le x \le a_{\frac{n+4}{2}} \\
Q_{\frac{n-4}{2}}'(x) = \frac{n-4}{n-2} - \frac{2}{n-2}\left(\frac{x-a_{\frac{n+4}{2}}}{a_{\frac{n+6}{2}}-a_{\frac{n+4}{2}}}\right) & \text{if } a_{\frac{n+4}{2}} \le x \le a_{\frac{n+6}{2}} \\
\quad \ldots\ldots\ldots\ldots\ldots\ldots & \\
\quad \ldots\ldots\ldots\ldots\ldots\ldots & \\
\quad \ldots\ldots\ldots\ldots\ldots\ldots & \\
Q_4'(x) = \frac{8}{n-2} - \frac{2}{n-2}\left(\frac{x-a_{n-4}}{a_{n-3}-a_{n-4}}\right) & \text{if } a_{n-4} \le x \le a_{n-3} \\
Q_3'(x) = \frac{6}{n-2} - \frac{2}{n-2}\left(\frac{x-a_{n-3}}{a_{n-2}-a_{n-3}}\right) & \text{if } a_{n-3} \le x \le a_{n-2} \\
Q_2'(x) = \frac{4}{n-2} - \frac{2}{n-2}\left(\frac{x-a_{n-2}}{a_{n-1}-a_{n-2}}\right) & \text{if } a_{n-2} \le x \le a_{n-1} \\
Q_1'(x) = \frac{2}{n-2}\left(\frac{a_n-x}{a_n-a_{n-1}}\right) & \text{if } a_{n-1} \le x \le a_n \\
0 & \text{if } x > a_n
\end{cases}
$$

Definition 3.4:

The α-cut of the polygonal fuzzy number $Q = (a_1, a_2, a_3, \ldots, a_n)$ (n is an even number) of the universe of discourse X is defined as $Q(\alpha) = \{x \in X / \mu_Q(x) \ge \alpha\}$ where $\alpha \in [0,1]$ where $Q_1(x), Q_2(x), \ldots, Q_{\frac{n-2}{2}}(x)$ are bounded and continuous increasing functions over $\left[0, \frac{2}{n-2}\right]$, $\left[\frac{2}{n-2}, \frac{4}{n-2}\right]$, $\left[\frac{4}{n-2}, \frac{6}{n-2}\right], \ldots, \left[\frac{n-4}{n-2}, 1\right]$ respectively and $Q_1'(x)$, $Q_2'(x), \ldots, Q_{\frac{n-2}{2}}'(x)$ are bounded and continuous decreasing functions over $\left[0, \frac{2}{n-2}\right], \left[\frac{2}{n-2}, \frac{4}{n-2}\right], \left[\frac{4}{n-2}, \frac{6}{n-2}\right], \ldots, \left[\frac{n-4}{n-2}, 1\right]$ respectively.

$$
Q(\alpha) = \begin{cases}
[Q_1(\alpha),\ Q_1'(\alpha)]; & \alpha \in \left[0, \frac{2}{n-2}\right) \\
[Q_2(\alpha),\ Q_2'(\alpha)]; & \alpha \in \left[\frac{2}{n-2}, \frac{4}{n-2}\right) \\
[Q_3(\alpha),\ Q_3'(\alpha)]; & \alpha \in \left[\frac{4}{n-2}, \frac{6}{n-2}\right) \\
\quad \ldots\ldots\ldots\ldots & \\
\quad \ldots\ldots\ldots\ldots & \\
\quad \ldots\ldots\ldots\ldots & \\
\left[Q_{\frac{n-1}{2}}(\alpha),\ Q_{\frac{n-1}{2}}'(\alpha)\right]; & \alpha \in \left[\frac{n-4}{n-2}, 1\right)
\end{cases}
$$

Then

$Q_1(x) = \alpha$ and $Q_1'(x) = \alpha$ gives the α-cut interval for $\alpha \in \left[0, \frac{2}{n-2}\right)$ as

$$\left[\frac{(n-2)}{2}(a_2 - a_1)\alpha + a_1,\ a_n - \frac{(n-2)}{2}(a_n - a_{n-1})\alpha\right]$$

$Q_2(x) = \alpha$ and $Q_2'(x) = \alpha$ gives the α-cut interval for $\alpha \in \left[\frac{2}{n-2}, \frac{4}{n-2}\right)$ as

$$\left[\left(\frac{(n-2)\alpha - 2}{2}\right)(a_3 - a_2) + a_2,\ \left(\frac{4-(n-2)\alpha}{2}\right)(a_{n-1} - a_{n-2}) + a_{n-2}\right]$$

$Q_3(x) = \alpha$ and $Q_3'(x) = \alpha$ gives the α-cut interval for $\alpha \in \left[\frac{4}{n-2}, \frac{6}{n-2}\right)$ as

$$\left[\left(\frac{(n-2)\alpha - 4}{2}\right)(a_4 - a_3) + a_3,\ \left(\frac{6-(n-2)\alpha}{2}\right)(a_{n-2} - a_{n-3}) + a_{n-3}\right]$$

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$Q_{\frac{n-1}{2}}(x) = \alpha$ and $Q_{\frac{n-1}{2}}'(x) = \alpha$ gives the α-cut interval for $\alpha \in \left[\frac{n-4}{n-2}, 1\right)$ as

$$\left[\left(\frac{(n-2)\alpha - (n-4)}{2}\right)\left(a_{\frac{n}{2}} - a_{\frac{n-2}{2}}\right) + a_{\frac{n-2}{2}},\ \left(\frac{(1-\alpha)(n-2)}{2}\right)\left(a_{\frac{n+4}{2}} - a_{\frac{n+2}{2}}\right) + a_{\frac{n+2}{2}}\right]$$

$\therefore\ Q(\alpha) =$

$$
\begin{cases}
\left[\frac{(n-2)}{2}(a_2 - a_1)\alpha + a_1,\ a_n - \frac{(n-2)}{2}(a_n - a_{n-1})\alpha\right]; & \alpha \in \left[0, \frac{2}{n-2}\right) \\
\left[\left(\frac{(n-2)\alpha-2}{2}\right)(a_3 - a_2) + a_2,\ \left(\frac{4-(n-2)\alpha}{2}\right)(a_{n-1} - a_{n-2}) + a_{n-2}\right]; & \alpha \in \left[\frac{2}{n-2}, \frac{4}{n-2}\right) \\
\left[\left(\frac{(n-2)\alpha-4}{2}\right)(a_4 - a_3) + a_3,\ \left(\frac{6-(n-2)\alpha}{2}\right)(a_{n-2} - a_{n-3}) + a_{n-3}\right]; & \alpha \in \left[\frac{4}{n-2}, \frac{6}{n-2}\right) \\
\quad \ldots\ldots\ldots\ldots & \\
\quad \ldots\ldots\ldots\ldots & \\
\quad \ldots\ldots\ldots\ldots & \\
\left[\left(\frac{(n-2)\alpha-(n-4)}{2}\right)\left(a_{\frac{n}{2}} - a_{\frac{n-2}{2}}\right) + a_{\frac{n-2}{2}},\ \left(\frac{(1-\alpha)(n-2)}{2}\right)\left(a_{\frac{n+4}{2}} - a_{\frac{n+2}{2}}\right) + a_{\frac{n+2}{2}}\right]; & \alpha \in \left[\frac{n-4}{n-2}, 1\right)
\end{cases}
$$

Example 3.2: Let $Q = (a_1, a_2, a_3, \ldots, a_{12})$ be a dodecagonal fuzzy number with membership function (figure 2.2)

$\mu_Q(x) =$

$$
\begin{cases}
0 & \text{if } x < a_1 \\
Q_1(x) = \frac{1}{5}\left(\frac{x-a_1}{a_2-a_1}\right) & \text{if } a_1 \le x \le a_2 \\
Q_2(x) = \frac{1}{5} + \frac{1}{5}\left(\frac{x-a_2}{a_3-a_2}\right) & \text{if } a_2 \le x \le a_3 \\
Q_3(x) = \frac{2}{5} + \frac{1}{5}\left(\frac{x-a_3}{a_4-a_3}\right) & \text{if } a_3 \le x \le a_4 \\
Q_4(x) = \frac{3}{5} + \frac{1}{5}\left(\frac{x-a_4}{a_5-a_4}\right) & \text{if } a_4 \le x \le a_5 \\
Q_5(x) = \frac{4}{5} + \frac{1}{5}\left(\frac{x-a_5}{a_6-a_5}\right) & \text{if } a_5 \le x \le a_6 \\
1 & \text{if } a_6 \le x \le a_7 \\
Q_5'(x) = 1 - \frac{1}{5}\left(\frac{x-a_7}{a_8-a_7}\right) & \text{if } a_7 \le x \le a_8 \\
Q_4'(x) = \frac{4}{5} - \frac{1}{5}\left(\frac{x-a_8}{a_9-a_8}\right) & \text{if } a_8 \le x \le a_9 \\
Q_3'(x) = \frac{3}{5} - \frac{1}{5}\left(\frac{x-a_9}{a_{10}-a_9}\right) & \text{if } a_9 \le x \le a_{10} \\
Q_2'(x) = \frac{2}{5} - \frac{1}{5}\left(\frac{x-a_{10}}{a_{11}-a_{10}}\right) & \text{if } a_{10} \le x \le a_{11} \\
Q_1'(x) = \frac{1}{5}\left(\frac{a_{12}-x}{a_{12}-a_{11}}\right) & \text{if } a_{11} \le x \le a_{12}
\end{cases}
$$

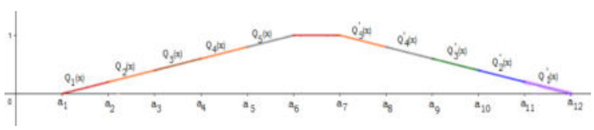

Figure 2.2

Dodecagonal fuzzy number

$Q_1(x), Q_2(x), Q_3(x), Q_4(x)$ and $Q_5(x)$ are bounded and continuous increasing functions over [0,0.2], [0.2,0.4], [0.4,0.6], [0.6,0.8] and [0.8,1.0] respectively.

$Q_1'(x)$, $Q_2'(x)$, $Q_3'(x)$, $Q_4'(x)$ and $Q_5'(x)$ are bounded and continuous decreasing functions over [0,0.2], [0.2,0.4], [0.4,0.6], [0.6,0.8] and [0.8,1.0] respectively.

Then its α-cut, $Q(\alpha)$ is given by ,

$$Q(\alpha) = \begin{cases} [a_1 + 5\alpha(a_2 - a_1),\ a_{12} - 5\alpha(a_{12} - a_{11})]; & \alpha \in [0,0.2) \\ [a_2 + (5\alpha - 1)(a_3 - a_2),\ a_{10} + (2 - 5\alpha)(a_{11} - a_{10})]; & \alpha \in [0.2,0.4) \\ [a_3 + (5\alpha - 2)(a_4 - a_3),\ a_9 + (3 - 5\alpha)(a_{10} - a_9)\ ]; & \alpha \in [0.4,0.6) \\ [a_4 + (5\alpha - 3)(a_5 - a_4),\ a_8 + (4 - 5\alpha)(a_9 - a_8)\ ]; & \alpha \in [0.6,0.8) \\ [a_5 + (5\alpha - 4)(a_6 - a_5),\ a_7 + (5 - 5\alpha)(a_8 - a_7)\ ]; & \alpha \in [0.8,1) \end{cases}$$

Definition 3.5: Arithmetic Operations on Polygonal Fuzzy Number with n vertices

Here basic arithmetic operations; addition, subtraction, multiplication and division are discussed by means of α-cut. Then arithmetic mean of polygonal fuzzy numbers is also considered. For defining basic operations, we consider two fuzzy numbers

$P_1 = (a_1, a_2, a_3, \dots, a_n)$ and $P_2 = (b_1, b_2, b_3, \dots, b_n)$.

(i) Addition: $P_1 + P_2 = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots, a_n + b_n)$

(ii) Subtraction: $P_1 - P_2 = (a_1 - b_n, a_2 - b_{n-1}, a_3 - b_{n-2}, \dots, a_n - b_1\ )$

(iii) Multiplication: $P_1 \times P_2 = (a_1 \times b_1, a_2 \times b_2, a_3 \times b_3, \dots, a_n \times b_n)$

(iv) Division: $P_1/P_2 = (a_1/b_1, a_2/b_2, a_3/b_3, \dots, a_n/b_n)$, $b_i \neq 0; i = 1,2 \dots, n$

(v) Arithmetic mean: $\frac{P_1+P_2}{2} = \left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2}, \dots, \frac{a_n+b_n}{2}\right)$

As for any $\alpha \in [0,1]$ the arithmetic intervals corresponding to the α-cuts are the same, $(P_1 + P_2)(\alpha) = P_1(\alpha) + P_2(\alpha)$

$(P_1 - P_2)(\alpha) = P_1(\alpha) - P_2(\alpha)$

$(P_1 \times P_2)(\alpha) = P_1(\alpha) \times P_2(\alpha)$

$(P_1/P_2)(\alpha) \quad = P_1(\alpha)/P_2(\alpha)$

We can confirm this using an example.

Consider two hendecagonal fuzzy numbers P_1 and P_2;

$P_1 = (0.2, 0.22, 0.24, 0.26, 0.28, 0.3, 0.32, 0.34, 0.36, 0.38, 0.4)$ and

$P_2 = (0.3, 0.33, 0.36, 0.39, 0.42, 0.45, 0.48, 0.51, 0.54, 0.57, 0.6)$

$P_1(\alpha) = [0.2 + 0.1\alpha, 0.4 - 0.1\alpha]$ and $P_2(\alpha) = [0.3 + 0.15\alpha, 0.6 - 0.15\alpha]$

for all $\alpha \in [0,1]$

(i) $P_1 + P_2 = (0.5, 0.55, 0.6, 0.65, 0.7, 0.75, 0.8, 0.85, 0.9, 0.95, 1.0)$

As for $\alpha \in [0,0.2)$, $\alpha \in [0.2,0.4)$, $\alpha \in [0.4,0.6)$, $\alpha \in [0.6,0.8)$ and $\alpha \in [0.8,1.0)$ the arithmetic intervals corresponding to the α-cuts are the same,

$(P_1 + P_2)(\alpha) = P_1(\alpha) + P_2(\alpha) = [0.5 + 0.25\alpha, 1 - 0.25\alpha]$ for all $\alpha \in [0,1]$

When $\alpha = 0$, $P_1(\alpha) + P_2(\alpha) = [0.5, 1]$

$\alpha = 0.2$, $P_1(\alpha) + P_2(\alpha) = [0.55, 0.95]$

$\alpha = 0.4$, $P_1(\alpha) + P_2(\alpha) = [0.6, 0.9]$

$\alpha = 0.6$, $P_1(\alpha) + P_2(\alpha) = [0.65, 0.85]$

$\alpha = 0.8$, $P_1(\alpha) + P_2(\alpha) = [0.7, 0.8]$

$\alpha = 1$, $P_1(\alpha) + P_2(\alpha) = [0.75, 0.75]$

This implies addition of alpha cuts lies within the interval

Hence the addition, $P_1 + P_2 = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots, a_n + b_n)$ is confirmed.

(ii) $P_1 - P_2 = (-0.4, -0.35, -0.3, -0.25, -0.2, -0.15, -0.1, -0.05, 0, 0.05, 0.1)$

As for $\alpha \in [0,0.2)$, $\alpha \in [0.2,0.4)$, $\alpha \in [0.4,0.6)$, $\alpha \in [0.6,0.8)$ and $\alpha \in [0.8,1.0)$ the arithmetic intervals corresponding to the α-cuts are the same,

$(P_1 - P_2)(\alpha) = P_1(\alpha) - P_2(\alpha) = [-0.4 + 0.25\alpha, 0.1 - 0.25\alpha]$ for all $\alpha \in [0,1]$

When $\alpha = 0$, $P_1(\alpha) - P_2(\alpha) = [-0.4, 0.1]$

$\alpha = 0.2$, $P_1(\alpha) - P_2(\alpha) = [-0.35, 0.05]$

$\alpha = 0.4$, $P_1(\alpha) - P_2(\alpha) = [-0.3, 0]$

$\alpha = 0.6$, $P_1(\alpha) - P_2(\alpha) = [-0.25, -0.05]$

$\alpha = 0.8$, $P_1(\alpha) - P_2(\alpha) = [-0.2, -0.1]$

$\alpha = 1$, $P_1(\alpha) - P_2(\alpha) = [-0.15, -0.15]$

This implies subtraction of alpha cuts lies within the interval

Hence the subtraction, $P_1 - P_2 = (a_1 - b_n, a_2 - b_{n-1}, a_3 - b_{n-2}, \dots, a_n - b_1\ )$ is confirmed.

(iii) $P_1 \times P_2 =$ $(0.06, 0.073, 0.086, 0.101, 0.118, 0.135, 0.154, 0.173, 0.194, 0.217, 0.24)$

As for $\alpha \in [0,0.2)$, $\alpha \in [0.2,0.4)$, $\alpha \in [0.4,0.6)$, $\alpha \in [0.6,0.8)$ and $\alpha \in [0.8,1.0)$ the arithmetic intervals corresponding to the α-cuts are the same,

$(P_1 \times P_2)(\alpha) = P_1(\alpha) \times P_2(\alpha) = [0.015\alpha^2 + 0.06\alpha + 0.06, 0.015\alpha^2 - 0.12\alpha + 0.24]$ for all $\alpha \in [0,1]$

When $\alpha = 0$, $P_1(\alpha) \times P_2(\alpha) = [0.06,\ 0.24]$

$\alpha = 0.2$, $P_1(\alpha) \times P_2(\alpha) = [0.073, 0.217]$

$\alpha = 0.4$, $P_1(\alpha) \times P_2(\alpha) = [0.086, 0.194]$

$\alpha = 0.6$, $P_1(\alpha) \times P_2(\alpha) = [0.101, 0.173]$

$\alpha = 0.8$, $P_1(\alpha) \times P_2(\alpha) = [0.118, 0.154]$

$\alpha = 1$, $P_1(\alpha) \times P_2(\alpha) = [0.135, 0.135]$

This implies multiplication of alpha cuts also lies within the interval.

Hence the multiplication, $P_1 \times P_2 = (a_1 \times b_1, a_2 \times b_2, a_3 \times b_3, \dots, a_n \times b_n\ )$ is justified.

(iv) Similarly, division $P_1/P_2 = (a_1/b_1, a_2/b_2, a_3/b_3, \dots, a_n/b_n\ )$ is justified with

$$(P_1/P_2)(\alpha) = \frac{P_1(\alpha)}{P_2(\alpha)} = \left[\frac{0.2 + 0.1\alpha}{0.3 + 0.15\alpha}, \frac{0.4 - 0.1\alpha}{0.6 + 0.15\alpha}\right]$$

(v) Arithmetic Mean is confirmed with

$$\left(\frac{P_1 + P_2}{2}\right)(\alpha) = \frac{P_1(\alpha) + P_2(\alpha)}{2} = [0.25 + 0.125\alpha, 0.5 - 0.125\alpha]$$

4. CONCLUSION

The formulas derived for finding membership function associated with any kind of fuzzy number and corresponding arithmetic operations are very useful in using fuzzy logic in different areas such as multi-criteria decision-making models, fuzzy optimization etc.

5. REFERENCES

[1] Ban, Adrian I., and Lucian Coroianu. "Existence, uniqueness and continuity of trapezoidal approximations of fuzzy numbers under a general condition." *Fuzzy sets and Systems* 257 (2014): 3-22.

[2] KIir, G.J.andYuan,B.,"Fuzzy Sets and Fuzzy Logic: Theory and Applications", Prentice Hall, New Delhi,(2001).

[3] Skalna, Iwona, et al. "Advances in fuzzy decision making." Studies in Fuzziness and Soft Computing. Springer, New York (2015).