RESEARCH PAPER | Engineering | Volume - 5 | Issue - 1 | Jan Special Issue - 2015 | ISSN - 2249-555X

Application of the Laplace Transform and Diffential Equation for RLC Circuit

KEYWORDS: Laplace transforms, Differential equation, RLC electrical circuit

Chirag R Patel
Asst. Professor, H&SC Dept.,SPBPEC, Saffrony, Mehsana

Shailesh T Patel
Asst. Professor and Head, H&SC Dept., SPBPEC,Saffrony, Mehsana

Vijay Makwana
Asst. Professor, H&SC Dept., GEC, Patan

Vijay Soni
Asst. Professor, H&SC Dept., GEC, Patan

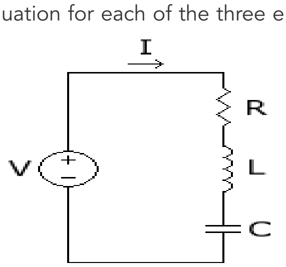
ABSTRACT Laplace transform is a very powerful mathematical tool applied in various areas of engineering and science. This paper presents an overview of the Laplace transform along with its application to several well-known in RLC electrical circuits. We obtain the solution of a fractional differential equation associated with a RLC electrical circuit

INTRODUCTION

B efore we examine the Laplace transform, let's rest discuss notation and then a problem related to the Laplace transform.Fractional differential equations have attracted in the recent years a considerable interest due to their frequent appearance in various field and their more accurate models of systems under consideration provided by fractional derivatives. For example, fractional derivatives have been used successfully to model frequency dependent damping behavior of many viscoelastic materials. They are also used in modeling of many chemical processes, mathematical biology and many other problems in Physics and Engineering. In this connection, one can refer to the monographs by Hilfer [5], Kilbas et al. [6], Kiryakova [7], Podlubny [9] and the various works cited therein. Debnath [2-4] considered solutions of fractional order homogeneous and non-homogeneous differential equations and integral equations in fluid mechanics. Magin and Ovadia [8] proposes modeling the cardiac tissue electrode interface using fractional calculus by means of a convenient three element electrical circuit. Camargo et al. [1] discuss the so-called telegraph equation in a fractional version whose solution is given in terms of a three-parameter Mittag-Leffler function and present also two new theorems involving the two and three parameter Mittag-Leffler functions. In a recent paper Soubhia et al. [10] studied a theorem involving series in the three-parameter Mittag-Leffler function and obtained the solution of a fractional differential equation associated with a RLC electrical circuit by the application of Laplace transform.

1.1Series RLC circuit

this circuit, the three components are all in series with the voltage source. The governing differential equation can be found by substituting into Kirchhoff's voltage law (KVL) the constitutive equation for each of the three elements. From KVL,



RLC series circuit

V - the voltage of the power source
I - the current in the circuit
R - the resistance of the resistor
L - the inductance of the inductor
C - the capacitance of the capacitor

$$v_R + v_L + v_C = v(t)$$

where v_R, v_L, v_C are the voltages across R, L and C is the time varying voltage from the source. Substituting in $v(t)$ is the constitutive equations,

$$Ri(t) + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{\tau=t} i(\tau)\, d\tau = v(t)$$

For the case where the source is an unchanging voltage, differentiating and dividing by L leads to the second order differential equation:

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = 0$$

This can usefully be expressed in a more generally applicable form:

$$\frac{d^2 i(t)}{dt^2} + 2\alpha\frac{di(t)}{dt} + \omega_0{}^2 i(t) = 0$$

α and ω_0 are both in units of angular frequenc α is called the neper frequency, or attenuation, and is a measure of how fast the transient response of the circuit will die away after the stimulus has been removed. Neper occurs in the name because the units can also be considered to be nepers per second, neper being a unit of attenuation. ω_0 is the angular resonance frequency. For the case of the series RLC circuit these two parameters are given by:

$$\alpha = \frac{R}{2L} \text{ and } \omega_0 = \frac{1}{\sqrt{LC}}$$

A useful parameter is the damping factor ζ which is defined as the ratio of these two,

$$\zeta = \frac{\alpha}{\omega_0}$$

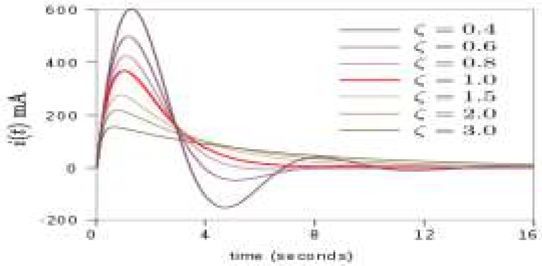
In the case of the series RLC circuit, the damping factor is given by,

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

The value of the damping factor determines the type of transient that the circuit will exhibit Some authors do not use ζ and call α the damping factor.

1.2 Transient response

Volume - 5 | Issue - 1 | Jan Special Issue - 2015 | ISSN - 2249-555X



Plot showing underdamped and overdamped responses of a series RLC circuit. The critical damping plot is the bold red curve. The plots are normalised for L=1, C=1 and ω0=1

The differential equation for the circuit solves in three different ways depending on the value of (ζ<1), overdamped (ζ>1),and critically .(ζ=1) The differential equation has the characteristic equation,[6]

$$s^2 + 2\alpha s + \omega_0{}^2 = 0$$

The roots of the equation in s are,[6]

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0{}^2}$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0{}^2}$$

The general solution of the differential equation is an exponential in either root or a linear superposition of both

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

The coefficients A1 and A2 are determined by the boundary conditions of the specific problem being analysed. That is, they are set by the values of the currents and voltages in the circuit at the onset of the transient and the presumed value they will settle to after infinite time.

1.3 Overdamped response

The overdamped response (ζ>1), is

$$i(t) = A_1 e^{-\omega_0(\zeta + \sqrt{\zeta^2 - 1})t} + A_2 e^{-\omega_0(\zeta - \sqrt{\zeta^2 - 1})t}$$

The overdamped response is a decay of the transient current without oscillation.

1.4 Underdamped response

The underdamped response **(** ζ<1) is,

$$i(t) = B_1 e^{-\alpha t} \cos(\omega_d t) + B_2 e^{-\alpha t} \sin(\omega_d t)$$

By applying standard trigonometric identities the two trigonometric functions may be expressed as a single sinusoid with phase shift,[11]

$$i(t) = B_3 e^{-\alpha t} \sin(\omega_d t + \varphi)$$

The underdamped response is a decaying oscillation at frequency ωd. The oscillation decays at a rate determined by the attenuation α. The exponential in α describes the envelope of the oscillation. B_1 and B_2 (or B_3 and the phase shift φ in the second form) are arbitrary constants determined by boundary conditions. The frequency ωd is given by,

$$\omega_d = \sqrt{\omega_0{}^2 - \alpha^2} = \omega_0 \sqrt{1 - \zeta^2}$$

This is called the damped resonance frequency or the damped natural frequency. It is the frequency the circuit will naturally oscillate at if not driven by an external source. The resonance frequency, ω0 , which is the frequency at which the circuit will resonate when driven by an external oscillation, may often be referred to as the undamped resonance frequency to distinguish it.

1.5 Critically Damped Response

The critically damped response (ζ=1)is,

$$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

The critically damped response represents the circuit response that decays in the fastest possible time without going into oscillation. This consideration is important in control systems where it is required to reach the desired state as quickly as possible without overshooting. D^1 and D^2 are arbitrary constants determined by boundary conditions.

2.1Defination of Laplace Transform

The Laplace Transform F(s) of f(t) is defined as

$$F(s) = \mathfrak{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

In this definition f(t) is assumed to be zero for t < 0. The Laplace variable s (p also used) is a complex variable which can take on all possible vluues. The Laplace Transform is well suited for describing systems with initial values and transients. F(s) is a complex function of a complex variable s. In the Fourier Transform, the transformed variable has a familiar meaning as a frequency spectrum. The Laplace variable s can be viewed as a generalisation of with the 2 being identical if the real part of s is restricted to a value of zero (ie the value of F(s) along the imaginary s axis is equivalent to the frequency spectrum of f(t). The fact that the general variable s can take on any complex value may seem intimidating, however as will be seen in many cases of interest only the values of s at the poles of F(s) are important in finding the inverse Laplace Transform. Not all f(t) have Laplace Transforms. The above integral does not converge for functions that increase faster than exponential $s = \sigma + i\omega$ e functions do not have a Laplace Transform. For many functions f(t) the transform only converges for some range of s eg for f(t) = e^{-at} the integral converges only for s > - a.

2.2Laplace domain

The series RLC can be analyzed for both transient and steady AC state behavior using the Laplace transform. f the voltage source above produces a waveform with Laplace-transformed *V(s)* (where s is the complex frequency

$$s = \sigma + i\omega),$$

KVL can be applied in the Laplace domain:

$$V(s) = I(s)\left(R + Ls + \frac{1}{Cs}\right)$$

where I(s) is the Laplace-transformed current through all components. Solving for *I(s):*

$$I(s) = \frac{1}{R + Ls + \frac{1}{Cs}} V(s)$$

And rearranging, we have that

$$I(s) = \frac{s}{L\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)} V(s)$$

2.3 Laplace admittance

Solving for the Laplace admittance *Y(s):*

$$Y(s) = \frac{I(s)}{V(s)} = \frac{s}{L\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

Simplifying using parameters α and ωo defined in the previous section, we have

$$Y(s) = \frac{I(s)}{V(s)} = \frac{s}{L(s^2 + 2\alpha s + \omega_0{}^2)}$$

2.4 Poles and zeros

The zeros of Y(s) are those values of s such that

$$Y(s) = 0,$$

$$s = 0 \quad \text{and} \quad |s| \to \infty$$

The poles of *Y(s)* are those values of s such that

$Y(s) \to \infty$ By the quadratic formula, we find

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0{}^2}$$

The poles of *Y(s)* are identical to the roots s_1 and s_2 of the characteristic polynomial of the differential equation in the section above.

2.5 General solution

For an arbitrary *E(t)*, the solution obtained by inverse transform of *I(s)* is:

$$I(t) = \frac{1}{L}\int_0^t E(t-\tau)e^{-\alpha\tau}\left(\cos\omega_d\tau - \frac{\alpha}{\omega_d}\sin\omega_d\tau\right)d\tau$$

in the underdamped case $(\omega_0 > \alpha)$

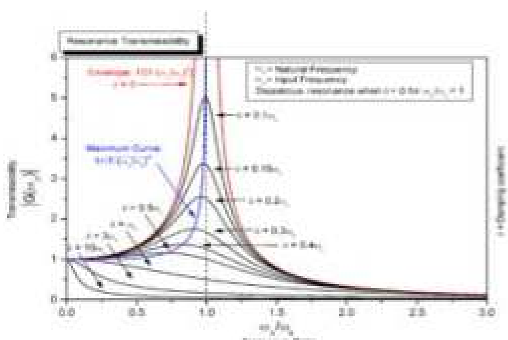
$$I(t) = \frac{1}{L}\int_0^t E(t-\tau)e^{-\alpha\tau}(1-\alpha\tau)d\tau$$

the critically damped case ($\omega_0=\alpha$)

$$I(t) = \frac{1}{L}\int_0^t E(t-\tau)e^{-\alpha\tau}\left(\cosh\omega_r\tau - \frac{\alpha}{\omega_r}\sinh\omega_r\tau\right)d\tau$$

where $\omega_r = \sqrt{\alpha^2 - \omega_0{}^2}$, and cosh and sinh are the usual hyperbolic functions.

2.6 Sinusoidal steady state



Steady state variation of amplitude with frequency and damping of a driven simple harmonic oscillator.

Sinusoidal steady state is represented by letting $s = i\omega$

Taking the magnitude of the above equation with this substitution:

$$|Y(s = i\omega)| = \frac{1}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

and the current as a function of ω can be found from

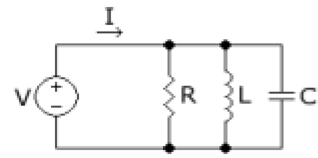
$$|I(i\omega)| = |Y(i\omega)||V(i\omega)|.$$

There is a peak value of $|I(i\omega)|$. The value of at this peak is *not* quite at the natural resonance frequency ω_0, (see diagram) although it approaches it when the damping is light.

The natural resonance frequency is: $\omega_0 = \frac{1}{\sqrt{LC}}.$

3.Parallel RLC circuit

The properties of the parallel RLC circuit can be obtained from the duality relationship of electrical circuits and considering that the parallel RLC is the dual impedance of a series RLC. Considering this it becomes clear that the differential equations describing this circuit are identical to the general form of those describing a series RLC.



RLC parallel circuit

V - the voltage of the power source
I - the current in the circuit
R - the resistance of the resistor
L - the inductance of the inductor
C - the capacitance of the capacitor

For the parallel circuit, the attenuation α is given by

$$\alpha = \frac{1}{2RC}$$

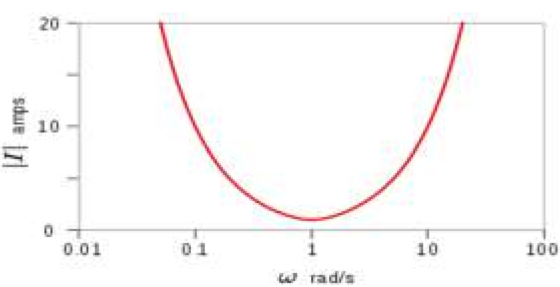
and the damping factor is consequently

$$\zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

This is the inverse of the expression for ζ in the series circuit. Likewise, the other scaled parameters, fractional bandwidth and Q are also the inverse of each other. This means that a wide band, low Q circuit in one topology will become a narrow band, high Q circuit in the other topology when constructed from components with identical values. The Q and fractional bandwidth of the parallel circuit are given by

$$Q = R\sqrt{\frac{C}{L}} \text{ and } F_b = \frac{1}{R}\sqrt{\frac{L}{C}}$$

3.1 Frequency domain



Sinusoidal steady-state analysis

normalised to *R* = 1 ohm, *C* = 1 farad, *L* = 1 henry, and *V* = 1.0 volt

The complex admittance of this circuit is given by adding up the admittances of the components:

$$\frac{1}{Z} = \frac{1}{Z_L} + \frac{1}{Z_C} + \frac{1}{Z_R} = \frac{1}{j\omega L} + j\omega C + \frac{1}{R}$$

The change from a series arrangement to a parallel arrangement results in the circuit having a peak in impedance at resonance rather than a minimum, so the circuit is an antiresonator.

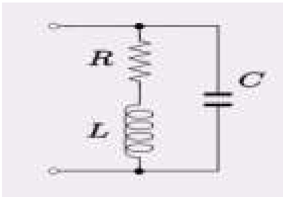
The graph opposite shows that there is a minimum in the frequency response of the current at the resonance frequency $\omega_0 = \frac{1}{\sqrt{LC}}$

when the circuit is driven by a constant voltage. On the other

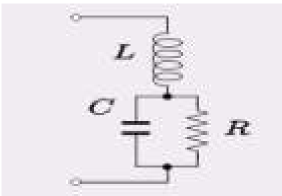
Volume - 5 | Issue - 1 | Jan Special Issue - 2015 | ISSN - 2249-555X

hand, if driven by a constant current, there would be a maximum in the voltage which would follow the same curve as the current in the series circuit.

3.2 Other configurations



RLC parallel circuit with resistance in series with the inductor



RLC series circuit with resistance in parallel with the capacitor

A series resistor with the inductor in a parallel LC circuit as shown in figure 7 is a topology commonly encountered where there is a need to take into account the resistance of the coil winding. Parallel LC circuits are frequently used for bandpass filtering and the Q is largely governed by this resistance. The resonant frequency of this circuit is,

$$\omega_0 = \sqrt{\frac{1}{LC} - \left(\frac{R}{L}\right)^2}$$

This is the resonant frequency of the circuit defined as the frequency at which the admittance has zero imaginary part. The frequency that appears in the generalised form of the characteristic equation (which is the same for this circuit as previously)

$$s^2 + 2\alpha s + {\omega'_0}^2 = 0$$

is not the same frequency. In this case it is the undamped resonant frequency $\omega'_0 = \sqrt{\frac{1}{LC}}$

The frequency ω_m at which the impedance magnitude is maximum is given by,

$$\omega_m = \omega'_0 \sqrt{\frac{-1}{Q_L^2} + \sqrt{1 + \frac{2}{Q_L^2}}}$$

where $Q_L = \frac{1}{R}\sqrt{\frac{L}{C}}$ is the quality factor of the coil. This can be well approximated by

$$\omega_m \approx \omega'_0 \sqrt{1 - \frac{1}{2Q_L^4}}$$

Furthermore, the exact maximum impedance magnitude is given by

$$|Z|_{max} = RQ_L^2 \sqrt{\frac{1}{2Q_L\sqrt{Q_L^2+2} - 2Q_L^2 - 1}}$$

For values of QL greater than unity, this can be well approximated by

$$|Z|_{max} \approx R\sqrt{Q_L^2(Q_L^2+1)}$$

In the same vein, a resistor in parallel with the capacitor in a series LC circuit can be used to represent a capacitor with a lossy dielectric. This configuration is shown in figure 8. The resonant frequency in this case is given by,

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{1}{(RC)^2}}$$

4.Conclusion :

This is also applicable in other circuit. And give solution laplace as well as ODE.

In particular, you should avoid using Word fonts or symbols for in-line single variables with superscripts or subscripts. Use italics for emphasis; do not underline. Turn off "smart quotes" (Tools | AutoCorrect | AutoFormat tabs). Turn off automatic hyphenation (Tools | Language | Hyphenation).

REFERENCE [1] Camargo, R.F., Chiacchio, A.O. and de Oliveira, E.C., 2008, "Differentiation to fractional orders and the telegraph equation", J. Math. Phys., 49, 033505. | [2] Debnath, L., 2003, "Fractional integrals and fractional differential equations in fluid mechanics", Fract. Calc. Appl. Anal., 6, pp. 119-155. | [3] Debnath, L., 2005, "Nonlinear Partial Differential Equations for Scientists and Engineers", Second ed., Birkhäuser, Basel. | [4] Debnath. L., 2003, "Recent Applications of Fractional Calculus to Science and Engineering", Internal, J. Math. and Math. Sci., 54, pp. 3413-3442. | [5] Hilfer (Ed.), R., 2000, "Applications of Fractional Calculus in Physics", World Scientific, Singapore.| [6] Kilbas, A.A., Srivastava, H.M., and Trujillo, J.J., 2006, "Theory and Applications of Fractional Differential Equations", Elsevier, Amsterdam. | [7] Kiryakova, V., 1994, "Generalized Fractional Calculus and Applications", John Wiley, New York. | [8] Magin R.L. and Ovadia, M., 2008, "Modeling the cardiac tissue electrode interface using fractional calculus", J. Vibration and Control, 14, pp. 1431-1442. | [9] Podlubny, I., 1999, "Fractional Differential Equations", Academic Press, San Diego. | [10] Soubhia, A.L., Camargo, R.F., de Oliveira, E.C. and Vaz Jr., J., 2001, "Theorem for series in three- parameter Mittag-Leffler function", Fract. Calc. Appl. Anal., 13, No.1, pp. 9-20.