Strong Convergence and Stability Result for New Multistep Iterative Scheme

Mathematics

KEYWORDS:

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ABSTRACT In this paper we introduce a new fixed point iterative scheme named CR multistep iterative scheme and prove strong convergence and stability for a pair of non self mappings using quasi contractive condition. Our result is generalization of some of the result in literature.

1. Introduction and Preliminary

Our main interest in this paper is to show strong convergence and stability of new multi step CR iterative schemes using quasi contractive operators. For the background of our exposition, we first mention some contractive mappings.

Zemfirescu [20] established a nice generalization of the Banach fixed point theorem as

Let T be a self map, there exists real numbers α, β, γ satisfying $0 \le \alpha < 1, 0 \le \beta < \frac{1}{2}, 0 \le \gamma < \frac{1}{2}$

respectively such that for each $x, y \in E$ at least one of the following is true:

$$\begin{aligned} &(z_1)\ d(Tx,Ty) \le \alpha\, d(x,y) \\ &(z_2)\ d(Tx,Ty) \le \beta [d(x,Tx) + d(y,Ty)] \\ &(z_3)\ d(Tx,Ty) \le \gamma [d(x,Ty) + d(y,Tx)] \end{aligned} \tag{1.1}$$

Then the mapping T satisfying (1.1) is called Zamfirescu contraction.

Remark 1.1: Mapping which satisfy (z_2) is called a Kannan mapping. While the mapping satisfying (z_3) is called Chatterjea operator.

The contractive condition (1.1) implies

$$\|Tx - Ty\| \le 2\delta \|x - Tx\| + \delta \|x - y\|, \forall\ x, y \in E, \tag{1.2}$$

Where

$$\delta = \max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\},\ 0 \le \delta < 1.$$

Rhoades [16, 17] used condition (1.1) in some convergence results for Mann and Ishikawa iteration process in a uniformly convex Banach space. Berinde [2] extended (1.1) to a new class of operators to an arbitrary Banach space and proved that this class is wider than class of Zamfirescu operators.

Several stability results are established by various authors in literature due to its increasing importance in computational mathematics. Some of the various authors whose contribution are much important in the study of stability of fixed point iteration process are Ostrowski [13], Harder and Hicks [6], Rhoades[15, 16], Osillike[12], Jachymski [7] ,Berinde [3,4] and Singh et al. [19]. The summability theory of infinite matrices are used by Harder and Hicks [6], Rhoades [16], Osilike[12], and Singh et al.[19] to prove different stability results for certain contractive conditions. The first stability result on T- stable mappings was due to Ostrowski[13] for the stability of Picard iteration using Banach contraction condition. Osilike and Udomene[15] introduced a shorter method for stability results and has also been employed by Berinde[3] ,Imoru and Olatinwo[10],Olatinwo et al. [11] and some others.

Now we recapitulate some iterative schemes in the literature of fixed point theory. Let X be a Banach space, Y an arbitrary set and $S, T: Y \to X$ such that $T(Y) \subseteq S(Y)$. For $x_0 \in Y$, the iterative scheme:

$$Sx_{n+1} = Tx_n, n = 0, 1, \tag{1.3}$$

was introduced by Jungck [8] in 1976 and it becomes the Picard iterative scheme when $S=I_d$ (identity mapping) and $Y=X$.

Singh et al. [19] defined the Jungck-Mann iterative scheme as

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tx_n, \tag{1.4}$$

was introduced by Jungck [8] in 1976 and it becomes the Picard iterative scheme when $S=I_d$ (identity mapping) and $Y=X$.

Singh et al. [19] defined the Jungck-Mann iterative scheme as

$$\begin{cases} Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Ty_n \\ Sy_n = (1-\beta_n)Sx_n + \beta_n Tx_n \end{cases} \tag{1.5}$$

and

$$\begin{cases} Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Ty_n \\ Sy_n = (1-\beta_n)Sx_n + \beta_n Tz_n \\ Sz_n = (1-\gamma_n)Sx_n + \gamma_n Tx_n \end{cases} \tag{1.6}$$

respectively, where $\{n\underline{a}\}$, $\{n\underline{b}\}$ and $\{n\underline{g}\}$ are sequences of positive numbers in [0,1].

Recently, Chugh and Kumar defined the Jungck-SP [5] iterative schemes as

$$\begin{cases} Sx_{n+1} = (1-\alpha_n)Sy_n + \alpha_n Ty_n \\ Sy_n = (1-\beta_n)Sz_n + \beta_n Tz_n \\ Sz_n = (1-\gamma_n)Sx_n + \gamma_n Tx_n \end{cases} \tag{1.7}$$

Olaleru and Akewe[21] defined the Jungck multistep Noor iteration as

$$\begin{cases} Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Ty_n^1 \\ Sy_n^i = (1-\beta_n^i)Sx_n + \beta_n^i Ty_n^{i+1}, i = 1,2....k-2 \\ Sy_n^{p-1} = (1-\beta_n^{k-1})Sx_n + \beta_n^{k-1} Tx_n \end{cases} \tag{1.8}$$

where $\{_n\underline{a}\}$ and $\{_{im}\underline{b}\}$ are sequences of positive numbers in [0,1].

Quite recently, Chugh and Kumar[14] introduced CR iterative scheme as

$$\begin{cases} Sx_{n+1} = (1-\alpha_n)Sy_n^1 + \alpha_n Ty_n^1 \\ Sy_n^i = (1-\beta_n^i)Sy_n^{i+1} + \beta_n^i Ty_n^{i+1}, i = 1,2...k-2 \\ Sy_n^{p-1} = (1-\beta_n^{k-1})Sx_n + \beta_n^{k-1} Tx_n \end{cases} \tag{1.9}$$

In this paper we use following iterative scheme

$$\begin{cases} Sx_{n+1} = (1-\alpha_n)Sy_n^1 + \alpha_n Ty_n^1 \\ Sy_n^i = (1-\beta_n^i)Tx_n + \beta_n^i Ty_n^{i+1}, i = 1,2....k-2 \\ Sy_n^{p-1} = (1-\beta_n^{k-1})Sx_n + \beta_n^{k-1} Tx_n \end{cases} \tag{1.11}$$

Remark 1.1 If X=Y and $S=I_d$ (identity mapping), then the Jungck multi step SP(1.9), Jungck multi step Noor(1.8), Jungck-SP (1.6), Jungck-Noor (1.5), Jungck CR(1.10), Jungck multi step CR(1.11), JungckIshikawa (1.3) and the Jungck-Mann (1.2) iterative schemes, respectively, become the multi step SP, multi step Noor, SP , Noor ,CR, multi step CR, Ishikawa and the Mann iterative schemes.

A more general contractive condition than (1.2) was used by Oltinwo[11] to prove stability and strong convergence in Jungck-Ishikawa iteration process.

(a) There exists a real number $a \in [0,1)$ and a monotone increasing function $\varphi : R^+ \to R^+$, such that $\varphi(0) = 0$ and $\forall\, x, y \in Y$, we have

$$\|Tx - Ty\| \le \varphi(\|Sx - Tx\|) + a\|Sx - Sy\| \tag{1.12}$$

(b) There exist a real number $M \ge 0$, $[0,1)a \in$ and a monotone increasing function $:\varphi : R^+ \to R^+$, such that $\varphi(0)0 =$ and $\forall$ x, y$\in$Y, we have

$$\|Tx-Ty\| \le \frac{\varphi(\|Sx-Tx\|)+a\|Sx-Sy\|}{1+M\|Sx-Tx\|} \qquad (1.13)$$

We use following definition and lemma

Definition 1.1. [10] Let S, T: Y→X such that $T(Y) \subseteq S(Y)$ and z a coincidence point of S and T, that is, $Sz = Tz = p$ (say). For any $x_0 \in Y$. Let the sequence $\{Sx_n\}_{n=1}^{\infty}$ generated by the iteration procedure (1.11) converges to p. Let $\{Sy_n\}_{n=0}^{\infty} \subseteq E$ be an arbitrary sequence, and set $\varepsilon_n = d(Sy_{n+1}, f(T, y_n)), \quad n == 0,1,2....$ Then, the iteration procedure (1.11) will be called (S, T)-stable if and only if $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} S(y_n) = p$.

Lemma 1.2. (Berinde [2]): If δ is a real number such that 0≤δ <1, and $\lim_{n\to\infty} \varepsilon_n = 0$ is a sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying

$$u_{n+1} \le \delta u_n + \varepsilon_n, \quad n = 0,1,2.....,$$

We have $\lim_{n\to\infty} u_n = 0$.|

Main results

2. STABILITY RESULT IN NORMED LINEAR SPACE

Theorem 2.1. Let $(E, \| \|)$ be a normed linear space and Y an arbitrary set. Suppose that S, T: Y→E are nonself operators such that T(Y) ⊆ S(Y). S(Y) a complete subspace of E and S is an injective operator. Let z be a coincidence point of S and T (that is, Sz=Tz=p). Suppose that S and T satisfy condition (1.12, 1.13). Let φ: $R^+ \to R^+$ be monotone increasing function such that φ (0) =0. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be Jungck CR multistep iteration process defined by (1.11) converging to p, where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n^i\}_{n=0}^{\infty}$ are sequences in [0, 1], such that 0< α ≤α_n and 0< β ≤ β_n^i. Then, the Jungck CR multi step iteration process is (S, T)-stable.

Proof:

Suppose that $\{Sx_n\}_{n=0}^{\infty} \subset E$, $\varepsilon_n = \|Sx_{n+1} - (1-\alpha_n)Sy_n^1 - \alpha_n Ty_n^1\|, n = 0,1,2..$ and let $\lim_{n\to\infty} \varepsilon_n = 0$. Then, we shall establish that $\lim_{n\to\infty} Sx_n = p$, using the contractive condition (1.12, 1.13) and the triangle inequality:

$$\begin{aligned}
\|Sx_{n+1} - p\| &\le \|Sx_{n+1} - (1-\alpha_n)Sy_n^1 - \alpha_n Ty_n^1\| + \\
&\qquad \|(1-\alpha_n)Sy_n^1 + \alpha_n Ty_n^1 - (1-\alpha_n+\alpha_n)p\| \\
&\le \|Sx_{n+1} - (1-\alpha_n)Sy_n^1 - \alpha_n Ty_n^1\| \\
&\qquad + (1-\alpha_n)\|Sy_n^1 - p\| + \alpha_n \|Ty_n^1 - p\| \\
&\le \varepsilon_n + (1-\alpha_n)\|Sy_n^1 - p\| + \alpha_n[\frac{\varphi(\|Sz-Tz\|)+a\|Sz-Sy_n^1\|}{1+M\|Sz-Tz\|}] \\
&\le \varepsilon_n + (1-\alpha_n(1-a))\|Sy_n^1 - p\|
\end{aligned} \qquad (2.1)$$

Application of (1.12, 1.13) and (1.11) gives

$$\begin{aligned}\left\|Sy_n^1 - p\right\| &= \left\|(1-\beta_n^1)Tx_n + \beta_n^1 Ty_n^2 + (1-\beta_n^1+\beta_n^1)p\right\| \\ &\le (1-\beta_n^1)\left\|Tx_n - p\right\| + \beta_n^1\left\|Ty_n^2 - p\right\| \\ &\le (1-\beta_n^1)a\left\|Sx_n - p\right\| + a\beta_n^1\left\|Sy_n^2 - p\right\|\end{aligned} \tag{2.2}$$

Using (2.2) in (2.1) yields

$$\left\|Sx_{n+1} - p\right\| \le \varepsilon_n + (1-\alpha_n(1-a))[(1-\beta_n^1)a]\left\|Sx_n - p\right\| + (1-\alpha_n(1-a))a\beta_n^1\left\|Sy_n^2 - p\right\| \tag{2.3}$$

Application of (1.12, 1.13) and (1.11) gives

$$\begin{aligned}\left\|Sy_n^2 - p\right\| &= \left\|(1-\beta_n^2)Tx_n + \beta_n^2 Ty_n^3 + (1-\beta_n^2+\beta_n^2)p\right\| \\ &\le (1-\beta_n^2)\left\|Tx_n - p\right\| + \beta_n^2\left\|Ty_n^3 - p\right\| \\ &\le (1-\beta_n^2)a\left\|Sx_n - p\right\| + a\beta_n^2\left\|Sy_n^3 - p\right\|\end{aligned} \tag{2.4}$$

Using (2.4) in (2.3) yields

$$\left\|Sx_{n+1} - p\right\| \le \varepsilon_n + (1-\alpha_n(1-a))[(1-\beta_n^1 a(1-a) - a^2\beta_n^1\beta_n^2)]\left\|Sx_n - p\right\| + a^2\beta_n^1\beta_n^2\left\|Sy_n^3 - p\right\| \tag{2.5}$$

Again application of (1.12, 1.13) and (1.11) gives

$$\begin{aligned}\left\|Sy_n^3 - p\right\| &= \left\|(1-\beta_n^3)Tx_n + \beta_n^3 Ty_n^4 + (1-\beta_n^3+\beta_n^3)p\right\| \\ &\le (1-\beta_n^3)\left\|Tx_n - p\right\| + \beta_n^3\left\|Ty_n^4 - p\right\| \\ &\le (1-\beta_n^3)a\left\|Sx_n - p\right\| + a\beta_n^3\left\|Sy_n^4 - p\right\|\end{aligned} \tag{2.6}$$

Using (2.6) in (2.5) yields

$$\begin{aligned}\left\|Sx_{n+1} - p\right\| \le \varepsilon_n + (1-\alpha_n(1-a))[(1-\beta_n^1 a(1-a) - a^2\beta_n^1\beta_n^2(1-a) - a^3\beta_n^1\beta_n^2\beta_n^3)\left\|Sx_n - p\right\| \\ + a^3\beta_n^1\beta_n^2\beta_n^3\left\|Sy_n^4 - p\right\|\end{aligned} \tag{2.7}$$

Continuing like this (2.7) yields

$$\begin{aligned}\left\|Sx_{n+1} - p\right\| \le \varepsilon_n + [(1-\alpha_n(1-a)][(1-\beta_n^1 a(1-a) - a^2\beta_n^1\beta_n^2(1-a)..-a^{k-3}\alpha_n\beta_n^1..\beta_n^{k-3})]\left\|Sx_n - p\right\| \\ + a^{k-3}\beta_n^1\beta_n^2..\beta_n^{k-3}\left\|Sy_n^{k-2} - p\right\|\end{aligned} \tag{2.8}$$

Using application of (1.12, 1.13) and (1.11) gives

$$\begin{aligned}\left\|Sy_n^{k-2} - p\right\| &= \left\|(1-\beta_n^{k-2})Tx_n + \beta_n^{k-2}Ty_n^{k-1} + (1-\beta_n^{k-2}+\beta_n^{k-2})p\right\| \\ &\le (1-\beta_n^{k-2})\left\|Tx_n - p\right\| + \beta_n^{k-2}\left\|Ty_n^{k-1} - p\right\| \\ &\le (1-\beta_n^{k-2})a\left\|Sx_n - p\right\| + a\beta_n^{k-2}\left\|Sy_n^{k-1} - p\right\|\end{aligned} \tag{2.9}$$

Using (2.9) in (2.8) yields

$$\begin{aligned}\left\|Sx_{n+1} - p\right\| \le \varepsilon_n + (1-\alpha_n(1-a))[(1-\beta_n^1 a(1-a) - a^2\beta_n^1\beta_n^2(1-a)..-a^{k-2}\alpha_n\beta_n^1..\beta_n^{k-2})]\left\|Sx_n - p\right\| \\ + a^{k-2}\beta_n^1\beta_n^2\beta_n^3..\beta_n^{k-2}\left\|Sy_n^{k-1} - p\right\|\end{aligned} \tag{2.10}$$

Using application of (1.12, 1.13) and (1.11) gives

$$\begin{aligned}\left\|Sy_n^{k-1}-p\right\| &\le \left\|(1-\beta_n^{k-1})Sx_n+\beta_n^{k-1}Tx_n+(1-\beta_n^{k-1}+\beta_n^{k-1})p\right\| \\ &\le (1-\beta_n^{k-1})\left\|Sx_n-p\right\|+\beta_n^{k-1}\left\|Tx_n-p\right\| \\ &\le (1-(1-a)\beta_n^{k-1})\left\|Sx_n-p\right\|\end{aligned} \tag{2.11}$$

Using (2.11) in (2.10) yields

$$\begin{aligned}&\left\|Sx_{n+1}-p\right\| \le \varepsilon_n+(1-\alpha_n(1-a))[(1-\beta_n^1 a(1-a)-a^2\beta_n^1\beta_n^2(1-a)..-a^{k-2}\beta_n^1\beta_n^2..\beta_n^{k-2}(1-a)\beta_n^{k-1})\left\|Sx_n-p\right\|] \\ &\le \varepsilon_n+(1-\alpha_n(1-a))(1-\beta_n^1 a(1-a))\left\|Sx_n-p\right\| \\ &\le \varepsilon_n+(1-\alpha_n(1-a))\left\|Sx_n-p\right\|\end{aligned}$$

Since $0 \le 1-(1-a)\alpha < 1$, using Lemma 2.1 in (2.11) yields

$\lim_{n\to\infty}\left\|Sx_n-p\right\|=0$, that is, $\lim_{n\to\infty}Sx_n=p$

Conversely, let $\lim_{n\to\infty}Sx_n=p$. Then, by using the triangle inequality and contractive definition, we have the following:

$$\begin{aligned}\varepsilon_n &= \left\|Sx_{n+1}-(1-\alpha_n)Sy_n^1-\alpha_n Ty_n^1\right\| \\ &\le \left\|Sx_{n+1}-p\right\|+\left\|(1-\alpha_n+\alpha_n)p-(1-\alpha_n)Sy_n^1-\alpha_n Ty_n^1\right\| \\ &\le \left\|Sx_{n+1}-p\right\|+(1-\alpha_n)\left\|Sy_n^1-p\right\|+\alpha_n\left\|Ty_n^1-p\right\| \\ &\le \left\|Sx_{n+1}-p\right\|+(1-\alpha_n(1-a))\left\|Sy_n^1-p\right\|\end{aligned} \tag{2.12}$$

Again, by the contractive condition (1.12, 1.13), we get

$$\begin{aligned}&\left\|Sy_n^1-p\right\| \le [(1-\beta_n^1 a(1-a)-a^2\beta_n^1\beta_n^2(1-a)..-a^{k-2}\beta_n^1\beta_n^2..\beta_n^{k-2}(1-a)\beta_n^{k-1})\left\|Sx_n-p\right\|] \\ &\le (1-\beta_n^1 a(1-a))\left\|Sx_n-p\right\|\end{aligned} \tag{2.13}$$

Hence, the iteration process defined in (1.11) is stable with respect to the pair (S, T).

3. STRONG CONVERGENCE IN AN ARBITRARY BANACH SPACE

Theorem 3.1: Let $(E,\|\ \|)$ be arbitrary Banach space and Y an arbitrary set. Suppose that S, T: Y⟶E are nonself operators such that T(Y) ⊆ S(Y). S(Y) a complete subspace of E and S is an injective operator. Let z be a coincidence point of S and T (that is, Sz=Tz=p). Suppose that S and T satisfy condition (1.12, 1.13). Let φ: $R^+ \to R^+$ be monotone increasing function such that φ (0) =0. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be Jungck CR multistep iteration process defined by (1.11) converging to p, where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n^i\}_{n=0}^{\infty}$ are sequences in [0, 1], such that $\sum_{k=0}^{\infty}\alpha_k=\infty$. Then, $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p.

Proof: Let C(S, T) be the set of the coincidence points of S and T. We shall now use condition (1.12) to establish that S and T have a unique coincidence point z (i.e. Sz=Tz=p (say)): Suppose that there exists $z_1, z_2 \in C(S, T)$ such that $Sz_1=Tz_1=p_1$ and $Sz_2=Tz_2=p_2$. If $p_1=p_2$, then $Sz_1=Sz_2$ and since S is injective, it follows that $z_1=z_2$.If $p_1 \neq p_2$, then we have by contractiveness condition (1.12) for S and T that

$$\begin{aligned}0 < \left\|p_1-p_2\right\| &= \left\|Tz_1-Tz_2\right\| \le \varphi(\left\|Sz_1-Tz_2\right\|)+a\left\|Sz_1-Sz_2\right\| \\ &= a\left\|p_1-p_2\right\|\end{aligned}$$

Which leads to $(1-a)\|p_1-p_2\|\le 0$, from which it follows that 1-a>0 since a∈ [0, 1), but $\|p_1-p_2\|\le 0$, which is a contradiction since norm is nonnegative. Therefore, we have that $\|p_1-p_2\|=0$, that is, $p_1=p_2=p$. Since $p_1=p_2$, then we have that $p_1=Sz_1=Tz_1==Sz_2=Tz_2=p_2$, leading to $Sz_1=Sz_2 \Rightarrow z_1=z_2=z$ (since S is injective). Hence, z∈ C(S, T), that is, z is unique coincidence point of S and T.

We now prove that $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p (where Sz=Tz=p) using again, condition (1.12, 1.13). Therefore, we have

$$\begin{aligned}\|Sx_{n+1}-p\| &= \|(1-\alpha_n)Sy_n^1+\alpha_nTy_n^1-(1-\alpha_n+\alpha_n)p\|\\ &\le (1-\alpha_n)\|p-Sy_n^1\|+\alpha_n\|p-Ty_n^1\|\\ &= (1-\alpha_n)\|p-Sy_n^1\|+\alpha_n\|Tz-Ty_n^1\|\\ &\le (1-\alpha_n)\|p-Sy_n^1\|+\alpha_n[\frac{\varphi(\|Sz-Tz\|)+a\|Sz-Sy_n^1\|}{1+M\|Sz-Tz\|}] \qquad (3.1)\\ &\le (1-\alpha_n)a\|Sy_n^1-p\|+\alpha_na\|Sy_n^1-p\|\\ &\le (1-\alpha_n(1-a)\|Sy_n^1-p\|\end{aligned}$$

By using application of (1.12, 1.13) and (1.11), we have

$$\begin{aligned}\|Sy_n^1-p\| &= \|(1-\beta_n^1)Tx_n+\beta_n^1Ty_n^2+(1-\beta_n^1+\beta_n^1)p\|\\ &\le (1-\beta_n^1)\|Tx_n-p\|+\beta_n^1\|Ty_n^2-p\|\\ &\le (1-\beta_n^1)a\|Sx_n-p\|+a\beta_n^1\|Sy_n^2-p\|\end{aligned}$$

(3.2)

$$\begin{aligned}\|Sy_n^2-p\| &= \|(1-\beta_n^2)Tx_n+\beta_n^2Ty_n^3+(1-\beta_n^2+\beta_n^2)p\|\\ &\le (1-\beta_n^2)\|Tx_n-p\|+\beta_n^2\|Ty_n^3-p\|\\ &\le (1-\beta_n^2)a\|Sx_n-p\|+a\beta_n^2\|Sy_n^3-p\|\end{aligned} \qquad (3.3)$$

$$\begin{aligned}\|Sy_n^3-p\| &= \|(1-\beta_n^3)Tx_n+\beta_n^3Ty_n^4+(1-\beta_n^3+\beta_n^3)p\|\\ &\le (1-\beta_n^3)\|Tx_n-p\|+\beta_n^3\|Ty_n^4-p\|\\ &\le (1-\beta_n^3)a\|Sx_n-p\|+a\beta_n^3\|Sy_n^4-p\|\end{aligned} \qquad (3.4)$$

$$\begin{aligned}\|Sy_n^{k-2}-p\| &= \|(1-\beta_n^{k-2})Tx_n+\beta_n^{k-2}Ty_n^{k-1}+(1-\beta_n^{k-2}+\beta_n^{k-2})p\|\\ &\le (1-\beta_n^{k-2})\|Tx_n-p\|+\beta_n^{k-2}\|Ty_n^{k-1}-p\|\\ &\le (1-\beta_n^{k-2})a\|Sx_n-p\|+a\beta_n^{k-2}\|Sy_n^{k-1}-p\|\end{aligned} \qquad (3.5)$$

Now using (3.2),(3.3),(3.4) and (3.5) in (3.1), we have

$$\begin{aligned}\|Sx_{n+1}-p\| &\le (1-\alpha_n(1-a))[(1-\beta_n^1a(1-a)-a^2\beta_n^1\beta_n^2(1-a)..-a^{k-2}\alpha_n\beta_n^1\beta_n^2..\beta_n^{k-2})]\|Sx_n-p\|\\ &\quad +a^{k-2}\beta_n^1\beta_n^2\beta_n^3..\beta_n^{k-2}\|Sy_n^{k-1}-p\|\end{aligned} \qquad (3.6)$$

$$\begin{aligned}\left\|Sy_n^{k-1}-p\right\| &\le \left\|(1-\beta_n^{k-1})Sx_n+\beta_n^{k-1}Tx_n+(1-\beta_n^{k-1}+\beta_n^{k-1})p\right\| \\ &\le (1-\beta_n^{k-1})\left\|Sx_n-p\right\|+\beta_n^{k-1}\left\|Tx_n-p\right\| \\ &\le (1-(1-a)\beta_n^{k-1})\left\|Sx_n-p\right\|\end{aligned} \tag{3.7}$$

Again using application of (1.12, 1.13) and (1.11), we have

$$\begin{aligned}\left\|Sx_{n+1}-p\right\| &\le (1-\alpha_n(1-a))[(1-\beta_n^1 a(1-a)-a^2\beta_n^1\beta_n^2(1-a)..-a^{k-2}\beta_n^1\beta_n^2..\beta_n^{k-2}(1-a)\beta_n^{k-1})\left\|Sx_n-p\right\|] \\ &\le (1-\alpha_n(1-a))(1-\beta_n^1 a(1-a))\left\|Sx_n-p\right\| \\ &\le (1-\alpha_n(1-a))\left\|Sx_n-p\right\| \\ &\le \prod_{k=0}^{n}[1-(1-a)\alpha_k]\left\|Sx_0-p\right\| \\ &\le \prod_{k=0}^{n}e^{-(1-a)\alpha_k}\left\|Sx_0-p\right\| \\ &\le e^{-[(1-a)\sum_{k=0}^{n}\alpha_k]}\left\|Sx_0-p\right\|\to 0 \text{ as } n\to\infty\end{aligned} \tag{3.8}$$

Since $\sum_{k=0}^{\infty}\alpha_k=\infty$ and a$\in$ [0, 1). Hence, we obtain from (3.8) that $\left\|Sx_n-p\right\|\to 0$ as $n\to\infty$, that is, $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p.

Remark 3.2. Since the iteration (1.10) is special cases of iterative scheme (1.11), so. Theorem 2.1 and Theorem 3 .1 generalizes existing result for (1.10) by taking k = 3.

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