

# Inverse Homotopy Perturbation Method for Nonlinear systems



## Mathematics

**KEYWORDS :** IHPM technique, boundary value problem, Klein-Gordon equation

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### ABSTRACT

*In this paper, a new analytical technique-Inverse Homotopy Perturbation Method (IHPM) for solving various ordinary/partial differential equations and integro-differential equations have been developed. A rule for selecting the best initial approximation has also been suggested. The IHPM is quite efficient and practically well suited for solving various problems involving differential and integro-differential equations. The comparison of the present method has also been carried out with other methods such as ADM, VIM and Taylor series method available in the literature for verification and validation purpose through some examples. The last section is devoted to numerical computations by using this technique and the corresponding results have been presented graphically.*

### 1.Introduction

Mathematical modeling of real-life problems usually results in functional equations, such as ordinary or partial differential equations, integral and integro-differential equations etc. These equations arise in various fields like fluid dynamics, solid mechanics, plasma physics, biological models and chemical kinetics. The solutions obtained from nonlinear wave equations are different from the solutions of the linear wave equations [1]. Now-a-days, it is a burning point for researchers to develop a better and more efficient method for nonlinear differential equations, integral and integro-differential equations which can give a better solution in an efficient manner by consuming less computational labor and time.

Adomian [2] proposed an analytical method known as adomian decomposition method (ADM) which is widely used to solve eminent nonlinear physical problems, linear and nonlinear differential equations, integral and integro-differential equations. In spite of good convergence, it is a difficult task to calculate adomian polynomials, especially in nonlinear problems. The Variational iteration method (VIM) proposed by He [3] has also been used by various authors rapidly involving the evaluation of Lagrange multiplier  $\lambda$  which becomes a difficult task in many cases. If, any how,  $\lambda$  is obtained, still in every iteration heavy computational work is required. The homotopy perturbation method (HPM) was introduced by He [4] which is actually a coupling of traditional perturbation method and homotopy method.

In order to overcome the extra computational labor and time, in this paper we proposed a new method, called Inverse Homotopy Perturbation Method (IHPM). It's supremacy over all existing methods can be realized by using many nonlinear differential equations and integro-differential equations.

### 2. Development of the method

To illustrate the basic concept of Inverse Homotopy Perturbation Method (IHPM), we consider the following nonlinear system of differential equations

$$A(u) = f(r), \quad r \in \Omega \tag{1}$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma$$

where  $A$  is a differential operator,  $B$  is a boundary operator, is  $f(r)$

a known analytic function and  $\Gamma$  is the boundary of the domain  $\Omega$ . Generally speaking the operator  $A$  can be divided into two parts, linear ( $L$ ) and nonlinear ( $N$ ). Thus the Eq. (1) can be written as [4]:

$$L(u) + N(u) - f(r) = 0$$

Upon using homotopy technique in topology, a homotopy is constructed as which satisfies the relation

$$\begin{aligned} v(r, p) : \Omega \times [0, 1] &\rightarrow R^n \\ H(v, p) &= (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \\ \text{OR } H(v, p) &= L(v) - (1-p)L(u_0) + p[N(v) - f(r)] = 0 \end{aligned}$$

Equivalently, the above homotopy relation can be rewritten as

$$L(v) = (1-p)L(u_0) - p[N(v) - f(r)] \tag{2}$$

where  $p \in [0, 1]$  is an embedding parameter and  $u_0$  is the best initial approximation of equation (1).

At  $p = 0$ , the system of equations is in sufficiently simplified form, and normally admits a rather simple solution. As  $p$  gradually increases to 1, the system follows a sequence of deformation, the solution of each of which is close to that at the previous stage of deformation. Eventually, at  $p = 1$ , the system takes the original form of equation and the final stage of deformation gives the desired solution. Therefore, Eq.(2) can be written as:

$$L(v) = L(u_0) - p[N(v) - f(r)] \tag{3}$$

Now applying the inverse operator  $L^{-1}$  to both sides of Eq. (3), we get

$$v = u_0 - p L^{-1}[N(v) - f(r)] \tag{4}$$

Using the homotopy parameter  $p$  as an expanding parameter, we have the following power series representation for  $v$ ,

$$v = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \dots = \sum_{i=0}^{\infty} p^i v_i$$

Hence Eq. (4) becomes

$$\sum_{i=0}^{\infty} p^i v_i = u_0 - p L^{-1}\left[N\left(\sum_{i=0}^{\infty} p^i v_i\right) - f(r)\right] \tag{5}$$

This is inverse homotopy perturbation method (IHPM) formula which can be used for illustrative problems in engineering and sciences.

The comparison of like powers of  $p$  leads to the evaluation of and hence  $v$ . This provides us the exact solution when  $p \rightarrow 1$

$$v_i, (i = 0, 1, 2, 3, \dots)$$

and thus we have

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + v_3 + \dots \tag{6}$$

The  $n^{th}$  order approximate solution to the Eq.(1) is given by

$$\Phi_n = v_0 + v_1 + v_2 + \dots + v_n \tag{7}$$

Remark 1: If  $u_0 = u(0) = 0$ , then  $L^{-1}$  [term with least power of  $r$  in  $f(r)$ ] should be taken as  $v_0$  for rapid convergence.

**3. Implementation of the Method**

In this section we outline the procedure for implementation of the inverse homotopy perturbation method to solve integro-differential equations, higher order boundary value problems and nonlinear partial differential equations:

**3.1. Integro-differential equations**

Consider the following integro-differential equation

$$y'(x) = r(x) + \int_a^b g(x)y^2(t)dt, \tag{7}$$

$$y(a) = \alpha \neq 0$$

Here, we take

$$L(y) = \frac{dy}{dx}, \quad N(y) = -\int_a^b g(x)y^2(t)dt$$

and applying IHPM formula (5), we have

$$\sum_{i=0}^{\infty} p^i v_i = \alpha + p L^{-1}[r(x)] + p L^{-1} \left[ \int_a^b g(x) \left( \sum_{i=0}^{\infty} p^i v_i \right)^2 \right] \tag{8}$$

Following the procedure of section 2, here we can obtain the required solution. In case of  $\alpha = 0$ , the best initial solution can be selected in light of remark 1.

**3.2. Higher Order Boundary Value Problems**

Consider the  $n^{th}$  order boundary value problem

$$y^{(n)}(x) = f(x) + h(x)y^r(x),$$

$$y^l(0) = \alpha_l, \quad l = 0, 1, 2, 3, \dots, n-1.$$

First of all, we convert it in the following form

$$y'(x) = g(x) + L^{-1}_{n-1}[f(x)] + L^{-1}_{n-1}[h(x)y^r(x)]$$

where  $g(x) = \sum_{j=1}^l y^{(j)}(0) \frac{x^{j-1}}{(j-1)!}$

Now, we write this as:

$$L(y) + N(y) = F(x)$$

where  $L(y) = \frac{dy}{dx}, \quad N(y) = -L^{-1}_{n-1}[h(x)y^r(x)], \quad F(x) = g(x) + L^{-1}_{n-1}[f(x)].$

Applying IHPM formula (5), we have

$$\sum_{i=0}^{\infty} p^i v_i = \alpha_0 + p L^{-1}[F(x)] + p L^{-1} \left[ \int_a^b g(x) \left( \sum_{i=0}^{\infty} p^i v_i \right)^r \right], \quad \alpha_0 \neq 0 \tag{9}$$

Proceeding in a similar manner as in section 2, required solution can be obtained. Again if  $\alpha_0 = 0$ , the initial solution can be selected as mentioned in remark 1.

**3.3. Partial Differential Equations**

Consider the following partial differential equation

$$u_t + c^2 u_x - \alpha u^2 = f(x,t)$$

$$u(x,0) = h(x), \quad u_t(x,0) = g(x).$$

First of all, we convert it in the following form

$$u_t(x,t) = F(x,t) - L_t^{-1}[c^2 u_x - \alpha u^2] \tag{10}$$

where  $F(x,t) = u_t(x,0) + L_t^{-1}[f(x,t)]$

Now we can rewrite Eq. (10) as

$$L(u) + N(u) = F(x,t)$$

where  $L(u) = u_t(x,t), \quad N(u) = -L_t^{-1}[c^2 u_x - \alpha u^2]$

Upon applying IHPM formula (5), we obtain

$$\sum_{i=0}^{\infty} p^i v_i = u(x,0) + p L^{-1}[f(x,t)] - p L^{-1} \left[ N \left( \sum_{i=0}^{\infty} p^i v_i \right) \right] \tag{11}$$

The Eq. (11) can directly be used as inverse homotopy perturbation method (IHPM) formula for partial differential equation. Further, the rest of the procedure is same as outlined in section 2. In case of  $u(x,0) = 0$ ,  $L^{-1}$  [term with least power of  $t$  in  $F(x,t)$ ] should be taken as  $v_0$  for rapid convergence as already mentioned in remark 1.

**4. Numerical Computations and Illustrations**

This section is devoted to verify the above method by solving various nonlinear problems and comparing the results with that obtained by other methods.

Illustration 1: We first consider the nonlinear integro-differential equation

$$u'(x) = -1 + \int_0^x u^2(t)dt \tag{12}$$

for  $x \in [0, 1]$  with the boundary condition  $u(0) = 0$ .

Using IHPM (8), along with remark 1, we have

$$\sum_{i=0}^{\infty} p^i v_i = L^{-1}[-1] + p L^{-1} \left[ \int_0^x \left( \sum_{i=0}^{\infty} p^i v_i \right)^2 dt \right] = -x + p L^{-1} \left[ \int_0^x (v_0^2 + 2v_0 v_1 p + 2v_1 v_1 + v_1^2 p^2 + \dots) dt \right]$$

Comparing terms of like powers of  $p$ , we obtain

$$p^0: v_0 = -x, \quad p^1: v_1 = \frac{x^4}{12}, \quad p^2: v_2 = -\frac{x^7}{252},$$

$$p^3: v_3 = \frac{x^{10}}{6048}, \quad p^4: v_4 = -\frac{x^{13}}{157248},$$

$$p^5: v_5 = \frac{37x^{16}}{158505984}, \dots$$

Clearly, the  $v_i, (i=0,1,2,\dots)$  can easily be calculated manually. The solution to Eq. (12) is given by

$$u(x) = \lim_{p \rightarrow \infty} v(x) = -x + \frac{x^4}{12} - \frac{x^7}{252} + \frac{x^{10}}{6048} - \frac{x^{13}}{157248} + \frac{37x^{16}}{158505984} + \dots \tag{13}$$

The approximate solution of fifth-order is written as:

$$\Phi_5(x) = -x + \frac{x^4}{12} - \frac{x^7}{252} + \frac{x^{10}}{6048} - \frac{x^{13}}{157248} + \frac{37x^{16}}{158505984} \tag{14}$$

The comparison of  $\Phi_5(x)$  for numerical results with 2-iterate VIM solution, ADM solution [5] and Taylor series solution [6] has been illustrated graphically in Fig. 1.

Illustration 2: Now we consider the next integro-differential equation

$$u'(x) = 1 + \int_0^x u^2(t) \frac{du(t)}{dt} \tag{15}$$

for  $x \in [0, 1]$  with the boundary condition  $u(0) = 0$ .

Using IHPM (8), along with remark 1, we have

$$\sum_{i=0}^{\infty} p^i v_i = L^{-1}[1] + p L^{-1} \left[ \int_0^x \left( \sum_{i=0}^{\infty} p^i v_i \right)^2 \left( \sum_{i=0}^{\infty} p^i v_i \right) dt \right] = x + p L^{-1} \left[ \int_0^x (v_0^2 + 2v_0 v_1 p + v_1^2 p^2 + 2v_0 v_1 + v_1^2 p + 2v_1 v_1 p^2 + \dots) dt \right]$$

Comparing terms of like powers of  $p$ , we obtain

$$p^0: v_0 = x, \quad p^1: v_1 = \frac{x^4}{12},$$

$$p^2: v_2 = \frac{x^7}{84}, \quad p^3: v_3 = \frac{19x^{10}}{10080}, \dots$$

Here, the third order approximate solution to Eq. (15) is given

by

$$\Phi_3(x) = v_0 + v_1 + v_2 + v_3 = x + \frac{x^4}{12} + \frac{x^7}{84} + \frac{19x^{10}}{10080} \quad (16)$$

The comparison of numerically computed results for  $\Phi_3(x)$  with 3-iterate VIM solution and ADM solution [5] has been presented in Fig. 2.

Illustration 3: Let us take the following nonlinear boundary value problem of fifth-order [7]

$$y^{(5)}(x) = e^{-x} y^2(x), \quad (17)$$

with boundary conditions

$$y(0) = y'(0) = y''(0) = 1, \quad y(1) = y'(1) = e.$$

The exact solution of this problem is  $y(x) = e^x$ .

Following the procedure given in section 3.2, we have

$$y'(x) = y'(0) + y''(0)x + y'''(0)\frac{x^2}{2} + y^{(4)}(0)\frac{x^3}{6} + L^{-1}[e^{-x}y^2(x)]$$

Now applying the IHPM formula (9), we obtain

$$\sum_{n=0}^{\infty} p^n v_n = 1 + pL^{-1}\left[1 + x + A\frac{x^2}{2} + B\frac{x^3}{6}\right] + pL^{-1}\left[e^{-x}\left(\sum_{n=0}^{\infty} p^n v_n\right)^2\right]$$

where  $y''(0) = A, y^{(4)}(0) = B$ .

Equivalently, we can write

$$\sum_{n=0}^{\infty} p^n v_n = 1 + p\left[x + \frac{x^2}{2} + A\frac{x^2}{6} + B\frac{x^3}{24}\right] + pL^{-1}\left[e^{-x}(v_0^2 + 2v_0v_1p + \dots)\right] \quad (18)$$

Comparing the coefficients of like powers of  $p$ , we have

$$p^0: v_0 = 1$$

$$p^1: v_1 = x + \frac{x^2}{2} + A\frac{x^2}{6} + B\frac{x^3}{24} + \int_0^x \int_0^{\xi_1} \int_0^{\xi_2} \int_0^{\xi_3} \int_0^{\xi_4} e^{-\xi} d\xi_4 d\xi_3 d\xi_2 d\xi_1 dx$$

$$= 1 + x^2 + \frac{(A-1)x^3}{6} + \frac{(B+1)x^4}{24} - e^{-x}$$

$$p^2: v_2 = \dots$$

Upon using Eq. (7), the 1st order approximate solution is written as

$$\Phi_1(x) = v_0 + v_1 = 2 + x^2 + \frac{(A-1)x^3}{6} + \frac{(B+1)x^4}{24} - e^{-x}$$

From the computer simulated results one can see that only first order approximate solution obtained by present method is very close to the exact solution.

Imposing the boundary conditions at  $x = 1$ , we get

$$\Phi_1(1) = 2 + 1 + \frac{(A-1)}{6} + \frac{(B+1)}{24} - e^{-1} = e$$

$$\Phi_1'(1) = 2 + \frac{(A-1)}{2} + \frac{(B+1)}{6} + e^{-1} = e$$

Solving these equations, we get

$$A = 0.9654561474, \quad B = 1.2060458815$$

Consequently, the approximate solution is given as:

$$\Phi_1(x) = 2 + x^2 + 0.0057573087x^3 + 0.0919185784x^4 - e^{-x}.$$

The comparison of computer simulated result of this approximate solution to the exact solution is given in Fig. 3.

**Example 4:** Now we consider the nonlinear Klein-Gordon equation [8, 9]:

$$u_t - u_x + u^2 = 2x^2 - 2t^2 + x^4 t^4, \quad u(x,0) = u_t(x,0) = 0.$$

The IHPM (11) here takes the form

$$\sum_{n=0}^{\infty} p^n v_n = L^{-1}\left[2x^2 t\right] + pL^{-1}\left[-2\frac{t^3}{3} + x^4 \frac{t^5}{5}\right]$$

$$+ pL^{-1}L^{-1}\left[v_0 + pv_1 + p^2v_2 + \dots\right] - (v_0 + pv_1 + p^2v_2 + \dots)^2$$

$$= x^2 t^2 + p\left[-\frac{t^4}{6} + x^4 \frac{t^6}{30}\right] + pL^{-1}\left[\int_0^t (v_0 - v_0^2) + (v_1 - 2v_0v_1)p + \dots\right] d\xi$$

Comparing coefficients of like powers of  $p$ , we obtain

$$p^0: v_0 = x^2 t^2$$

$$p^1: v_1 = -\frac{t^4}{6} + \frac{x^4 t^6}{30} + L^{-1}\left[\frac{2t^2}{3} - \frac{x^4 t^2}{5}\right] = -\frac{t^4}{6} + \frac{x^4 t^6}{30} + \frac{t^4}{6} - \frac{x^4 t^6}{30} = 0$$

$$p^2: v_2 = L^{-1}\left[\int_0^t (v_1 - 2v_0v_1) d\xi\right] = 0, \quad p^3: v_3 = 0, \dots$$

Thus

$$u(x,t) = \lim_{p \rightarrow 1} v(x,t) = v_0 + v_1 + v_2 + v_3 + \dots = x^2 t^2$$

which is the exact solution.

### 5. Results and Discussion

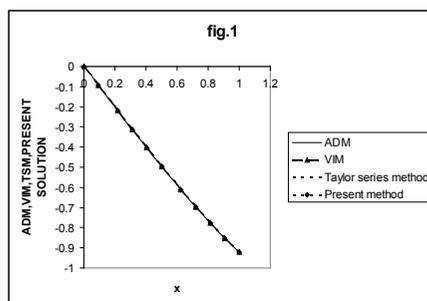
In order to discuss the entire results and development in this paper the approximate solutions of different orders in Illustrations 1, 2, and 3 have been obtained. The computer simulated results obtained by MathCAD software

are compared with results obtained by other methods such as Adomian decomposition method (ADM) and Variational iteration method (VIM) [5], Taylor series method [6]. The comparisons show that the present solution is better than the other solutions. In Illustration 3 the present solution is compared with exact solution. Although the present solution is the only first order approximation, it is very close to exact solution. Obviously, increasing the order of approximation one can increase the accuracy more and more. In Illustration 4, we have obtained the exact solution by taking the best initial approximation.

In this method we suggest a fixed rule for the best initial approximation while in other methods, there is no such definite rule. This best initial approximation gives a rapid convergence.

### 6. Conclusion

In this work, we have developed a new method-inverse homotopy perturbation method (IHPM) in which homotopy perturbation has not been used in a direct way. The IHPM is applicable to a variety of linear and nonlinear differential equations, integro-differential equations and boundary value problems. A fixed rule to choose the best initial approximation which leads to get rid of hit and trial initial approximations has also been suggested. This method gives more realistic series solutions leading to a rapid convergence and even an exact solution in many cases. It is worth mentioning that the IHPM is capable of in comparison to the other classical methods, in addition to maintaining the high accuracy of numerical results. It may be concluded that the IHPM is very powerful and efficient technique in finding analytical and numerical solutions for a wide variety of linear and nonlinear problems arising in different disciplines of engineering and sciences.



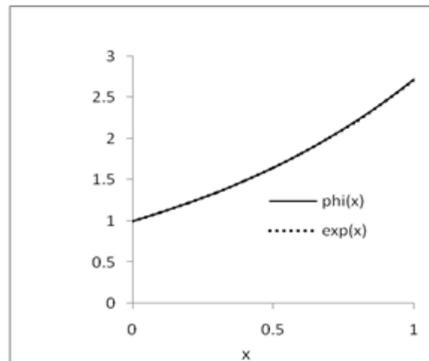
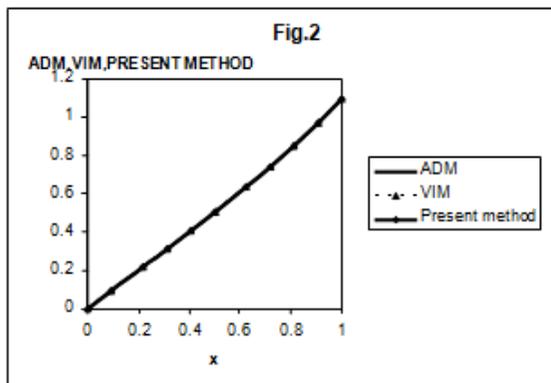


Fig.3: Comparison of present solution  $\phi(x)$  and exact solution  $\exp(x)$  of Illustration 3.

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