Research Paper

Volume : 2 | Issue : 2 | Feb 2013 • ISSN No 2277 - 8179

Mathematics

KEYWORDS : Uptake parameter, asymptotic matching, relative water content

Asymptotic Analysis of A Model for Salt Transport by The Root of A Plant

Arun Kumar Department of Mathematics, Government College, Kota (Raj.), India

ABSTRACT

A mathematical model for salt transport in a cylindrical root in the field soil is derived and solved analytically by asymptotic matching of the inner and outer solutions. Asymptotic analysis of the model shows that the solute uptake by a single cylindrical root in the absence of competition does not influence the water or solute concentration in the soil significantly in both saturated and unsaturated soil conditions.

1. Introduction:

Salt with water is taken up by the plant roots and transported to the stems and leaves along the xylem tubes of the root [1]. Xylem tubes are located in the central part of root inside the Caspian strip. It consists of several different types of cells, the most important ones are tracheary elements and fibers which are the primary source of mechanical strength. Tracheary elements are non-living cells [1] and are mainly involved in solute transport.

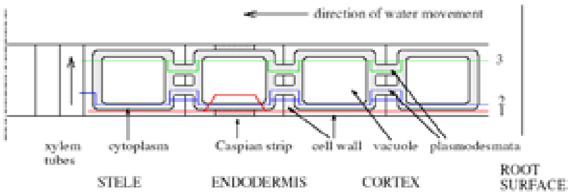

Figure 1: Radial solute movement pathways

For salt to be transported from soil to the plant leaves, it first has to move from the soil to the xylem, i.e., it has to pass through the cortex and endodermis of the root (Figures 1). The pathways along which it does so are still a matter of debate, but according to Oertli [2] and Steudle and Peterson [3] there are at least 3 main pathways: Line 1 describes the two possible apoplastic pathways, Line 2 describes two possible transcellular, i.e., vacuole pathways and Line 3 describes the two possible cytoplasmic pathways.

Solute flows in the pore system of the cell walls, i.e., in the apoplast and near the Caspian strip this pathway is blocked and hence it passes through the vacuole due to the resistivity of the Caspian strip There are two phenomenon in the process:

1. Solute flows inside the cells in the symplast and transfer from cell to cell occurs either through apoplast or plasmodesmata (Line 2).
2. Salt transport inside the cell occurs across the cytoplasm and then through vacuole (Line 3)

Whichever pathway dominates is strongly dependent on the environmental conditions the plant is growing in. For simplicity we will consider the root endodermis to be a porous material where all possible pathways are absorbed into one radial endodermal solute conductivity parameter.

In this paper we aim to develop a model for salt transport and uptake by plant roots and then incorporate this into the pollutant uptake model.

2. Salt transport in a single cylindrical root with finite length

Model derivation

The equation for the conservation for incompressible solute in the soil, is given by

$$\frac{\partial \phi_l}{\partial t} = -\nabla.u - F_w, \quad (1)$$

where ϕ_l is the volume of solute per unit volume of soil, u is the volume flux of solute and Fw is the volume uptake by plant roots per unit volume of soil and in the absence of solute sink

$$\frac{\partial \phi_l}{\partial t} + \nabla.u = 0 \quad (2)$$

The flux of solute u in the soil is generally given by Darcy's Law ([4], [5]) which states that the specific discharge (Darcy flux)

$$u = -\frac{k}{\mu}\left[\nabla p - \rho g \hat{k}\right], \quad (3)$$

where k is the soil permeability, μ is the viscosity of solute, and p is the solute pressure in the soil, and $\rho g \hat{k}$ represents the gravitational effects [10]. $\hat{k}$ is the unit vector pointing downwards from the soil surface and given by the van Genuchten formula [7] in terms of salt concentration S

$$k(S) = k_s S^{1/2}\left[1-\left(1-S^{1/m}\right)^m\right]^2 \quad (4)$$

$$for\ m = 1-\frac{1}{n},\ 0<m<1,$$

$$\text{and } S = \frac{\phi_l - \phi_{l,r}}{\phi_{l,s} - \phi_{l,r}} = \left[\frac{1}{1+\left(-\frac{\alpha}{\rho g}p\right)^n}\right]^m \quad (5)$$

where $\phi_{l,r}$ is the residual (minimum) solute concentration of the soil, and $\phi_{l,s}$ is the saturated solute concentration of the soil, i.e., the soil porosity, k_s is the hydraulic conductivity in fully saturated soil, α is a fitting parameter [6] that can be estimated directly from experimental suction characteristic data.

Writing the model in terms of S using the full van Genuchten formulas, modified by Clausnitzer and Hopmans etal. ([8],[9]), we get the following nonlinear diffusion-convection equation

$$\left(\phi_{l,s} - \phi_{l,r}\right)\frac{\partial S}{\partial t} = \nabla.(D(S)\nabla S) - \frac{\rho g}{\mu}\frac{dk}{dS}\frac{\partial S}{\partial z} \quad (6)$$

where D(S) is called the solute diffusivity in soil [7]and it is defined as

$$D(S) = \frac{k_s}{\mu}\left|\frac{\partial p}{\partial S}\right| = \frac{k_s \rho g}{\mu} \times \frac{(1-m)}{\alpha m} S^{1/2-1/m} \times[(1-S^{1/m})^{-m} + (1-S^{1/m})^m - 2], \quad (7)$$

and

$$\frac{dk(S)}{dS} = k_s\left[\frac{1}{2}S^{-1/2}[1-(1-S^{1/m})^m]^2 + 2S^{-1/2+1/m}[1-(1-S^{1/m})^m](1-S^{1/m})^{m-1}\right] \quad (8)$$

Parameter	ϕl,s	ϕl,r	$K_s = \frac{k_s \rho g}{\mu}$	α	N	ε_0	ε_1 $K_s = \frac{k_s \rho g}{\mu}$
Units	$cm^3\ cm^{-3}$	$cm^3\ cm^{-3}$	cm3 /day	cm^{-1}	----		
Sandstone	0.250	0.153	108.0	0.0079	10.5	0.266	5.98x10-7

Silt Loam	0.396	0.131 107423	4.96	0.00423	2.06	4.461	3.58x10-8
Clay Loam (drying)	0.446	0.0	0.82	0.00152	1.17	1.688	9.43x10-8

Table 1: Values for van Genuchten coefficients [7]

3. Approximate Solution

The flow of water in the soil using the Darcy- Genuchten model and taking into account the flow inside the root with a correction term for flow outside the root, can be obtained by writing equations (6) in terms of the cylindrical polar coordinate r, where $r=\sqrt{x^2+y^2}$

$$(\phi_{l,s}-\phi_{l,r})\frac{\partial S}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}(rD(S)\frac{\partial S}{\partial r}) \quad (9)$$
$$+\frac{\partial}{\partial z}(D(S)\frac{\partial S}{\partial z})-\frac{\rho g}{\mu}\frac{\partial k(S)}{\partial z}$$

Where

$$D(S)=\frac{k_s\rho g(1-m)}{\mu\alpha m}S^{1/2-1/m}$$
$$\times[(1-S^{1/m})^m+(1-S^{1/m})^{-m}-2], \quad (10)$$

and

$$k=k_sS^{1/2}\left[1-\left(1-S^{1/m}\right)^m\right]^2 \quad (11)$$
$$for\ m=1-\frac{1}{n},\ 0<m<1,$$

Root Radial Boundary Conditions

Far away from the root the concentration is not undisturbed by the root uptake, so

$$\frac{\partial S}{\partial r}\to 0\ \ as\ r\to 0$$

However, the boundary condition at the root surface needs to be modified to include the longitudinal variations in the root internal and external pressure. This can be determined by the, concentration of the solute in the root i.e solution of the equation (12) at r = a,

$$2\pi ak_r[p(r,z)-p_r(z)]=-k_x\frac{\partial^2 p_r}{\partial z^2}\ \ at\ r=a, \quad (12)$$

with boundary conditions

$$p_r=T\ \ at\ z=0,\ and\ \frac{\partial p_r}{\partial z}=0\ \ at\ \ \ z=L \quad (13)$$

The solution of equation (12) for pr(z) is given in terms of integrals of p(r, z) at r = a. Hence we will have a mixed differential-integral boundary condition at the root surface, i.e.

$$\frac{k(p)}{\rho g}\frac{\partial p}{\partial r}=k_r(p-p_r(z))\ \ at\ r=a, for\ \ 0\langle z\leq L \quad (14)$$

where L is the length of the root.

Non-dimensionalation:

Non-dimensionalising the z directional length with the maximum length of the root L, the r directional length with the root radius a, and choosing the time-scale [t] such that the time-derivative term balances the radial diffusion term, i.e., by choosing

$$z=Lz^*,\ r=ar^*,\ p=\frac{\rho g}{\alpha}p^*, \quad (15)$$

$$t=[t]t^*\ \ with\ [t]=\frac{(\phi_{l,s}-\phi_{l,r})\mu\alpha ma^2}{\rho gk_s(1-m)},$$

the dimensionless equation becomes (after dropping s)

$$\frac{\partial S}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}(rD(S)\frac{\partial S}{\partial r})+ \quad (16)$$
$$\in_1\frac{\partial}{\partial z}(D(S)\frac{\partial S}{\partial z}+\in_0\frac{\partial k}{\partial z}$$

where $\in_0=\frac{\alpha ma^2}{L(1-m)}$, $\in_1=\frac{(1-m)}{L\ \alpha m}$ (17)

$$D(S)=S^{1/2-1/m}[(1-S^{1/m})^m+(1-S^{1/m})^{-m}-2], \quad (18)$$

and

$$k=k_sS^{1/2}\left[1-\left(1-S^{1/m}\right)^m\right]^2 \quad (19)$$
$$for\ m=1-\frac{1}{n},\ 0<m<1,$$

The dimensionless boundary condition will be

$$D(S)\frac{\partial S}{\partial r}=\lambda_w[p(S)-p_r]\ at\ r=1,\ \ 0<z<1,$$
$$with\ p(S)=-(S^{-1/m}-1)^{1/n} \quad (20)$$

where dimensionless parameters λ_w and p_r are given by

$$\lambda_w=\frac{k_r\rho gma}{(1-m)k_s} \quad (21)$$

The far-field boundary condition is still

$$S\to S_\infty\ \ as\ r\to\infty, \quad (22)$$

The internal root pressure in terms of external root pressure is given as a solution to the dimensionless analogue of the equation (12) with boundary conditions (13), i.e by non dimensionalising

$$p_r=\frac{\rho g}{\alpha}p^*\ \ \ and\ \ p=\frac{\rho g}{\alpha}p_r^*$$

We get

$$-\frac{\partial}{\partial}=k\ \ [p(1,z)-p\ \ (z)] \quad (23)$$

$$with\ p\ \ =T\ \ at\ z=0\ \ and\ \frac{\partial}{\partial}=0\ \ at\ z=1 (24)$$

where $k=L\sqrt{2\pi ak_r/k_x}\approx 1$ for the root xylem and radial conductivity and $T_0=\frac{T\alpha}{\rho g}$

Hence, the solution in the Green's function form becomes

$$p_r(z)=T_0+k^2\int_0^1 G(z,\xi)[p(1,z)-T_0]d\xi \quad (25)$$

$$with\ \ G(z,\xi)\begin{cases}-\frac{1}{k}\sinh(kz)[c\operatorname{osh}(k\xi)- \\ \tanh(k)\sinh(k\xi)]\quad 0\leq z\leq\xi\leq 1 & (26)\\ -\frac{1}{k}\sinh(kz)[c\operatorname{osh}(kz)- \\ \tanh(k)\sinh(kz)]\quad 0\leq\xi\leq z\leq 1\end{cases}$$

We begin the analysis of this dimensionless model by considering the system at the leading order only, i.e., neglecting all O(ε0) terms, and assuming that as r→∞ the salt concentration is a constant that is independent of z and t, i.e., we take $S\to S_\infty$ as $r\to\infty$. In general the uptake coefficient $\lambda_w\sim O(10^{-7})$ and also $-\lambda_w p_r\sim O(10^{-5})$ are very small for common crops like wheat, maize and soybean. So there will be small salt concentration gradients in the soil due to uptake. So the problem to be solved becomes

$$\frac{\partial s}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(rD\left(S\right)\frac{\partial s}{\partial r}\right) \quad (27)$$

with

$$D(S)\frac{\partial s}{\partial r}=\lambda_w\left(p_1-p_r(p_1)\right)\ \ r=1, \quad (28)$$

and

$$S\to S_\infty\ \ as\ \ r\to\infty$$

$$-\frac{\partial^2 p_r}{\partial z^2}=k^2[p(1,z)-p_r(z)]with$$

and

$$p_r=T_0\ \ at\ \ z=0\ \ \ and\ \ \frac{\partial p_r}{\partial z}=0\ at\ \ z=1 \quad (29)$$

Volume : 2 | Issue : 2 | Feb 2013 • ISSN No 2277 - 8179

where p1 is the solute pressure at the surface of the root.

Outer Solution: We now rescale the outer region far away from the root, i.e., $r = R/\sigma$ and $t = \tau/\sigma^2$ with << 1. Taking into account that far away from the root the disturbance to the uniform concentration profile is small, i.e., s<< 1, we approximate the soil water diffusivity by expanding it in Taylor series and using the highest order term only. Hence we have

$$D(S_\infty + s) \approx D(S_\infty) + O(s). \qquad (30)$$

Therefore the highest order outer problem is

$$\frac{\partial s}{\partial \tau} = \frac{1}{R}\frac{\partial}{\partial R}(RD(S)\frac{\partial s}{\partial R}), \qquad (31)$$

with $s \to 0 \quad as \quad r \to \infty$ (32)

The outer similarity type solution is given by (using the similarity variable η= r2/(4t))

$$S = S_\infty + BE_1\left(\frac{r^2}{4tD(S_\infty)}\right), \qquad (33)$$

where B is the matching constant, which will be determined by matching this inner solution with the outer solution.

Inner Solution: Rescaling back in space to the inner region near the root the inner problem becomes

$$\sigma^2 \frac{\partial S}{\partial \tau} = \frac{1}{r}\frac{\partial}{\partial r}(rD(S_\infty)\frac{\partial S}{\partial r}), \qquad (34)$$

with

$$\frac{\partial S}{\partial \tau} = \lambda_m \frac{p(S_\infty + s)}{D(S_\infty + s)} - \lambda_w \frac{p_r}{D(S_\infty + s)} \quad at \quad r = 1, \qquad (35)$$

$$\approx \frac{\lambda_w}{D}(p(S_\infty) - p_r) + \frac{p'D - D'(p - p_r)}{D^2}\Big|_{S_\infty} s + O(s^2) \quad r = 1 \qquad (36)$$

$$\approx \frac{\lambda_w}{D}(p_1 - p_r(p_1)) + \frac{p'D - D'(p - p_r)}{D^2}\Big|_{S_\infty} s + O(s^2) \quad r = 1$$

where $D = D(S_\infty)$ and . $D' = \partial D(S)/\partial S|_{s=S_\infty}$

Hence, as long as $S(1,t) \sim S_\infty$ we have $|s| << S_\infty$, therefore the leading order inner problem is

$$\frac{1}{r}\frac{\partial}{\partial r}(rD(S_\infty)\frac{\partial s}{\partial r}) = 0, \qquad (37)$$

with the leading order boundary condition corresponding to constant flux, i.e.,

$$\frac{\partial s}{\partial r} = \frac{\lambda_w}{D(S_\infty)}(p_1 - p_r(p_1)) \quad at \ r = 1. \qquad (38)$$

The leading order inner solution is therefore given by

$$S = S_\infty + s_1 + \frac{\lambda_w[(p_1 - p_r(p_1))]}{D(S_\infty)} lnr \qquad (39)$$

Matching: To match the leading order inner solution to the leading order outer solution we expand the exponential integral for small argument we find that the outer solution becomes

$$S \approx S_\infty + B\{-2ln\,r + ln[4e^{-\gamma}D(S_\infty)t] + \ldots\ldots\} \qquad (40)$$

Matching O(ln r) terms in the inner and outer expansion gives that

$$B = -\frac{\lambda_w[(p_1 - p_r(p_1))]}{2D(S_\infty)}, \qquad (41)$$

and matching O(1) terms gives

$$S_1 = S_\infty - \frac{\lambda_w[(p_1 - p_r)]}{2D(S_\infty)} ln[4e^{-\gamma}D(S_\infty)t + 1], \qquad (42)$$

Where S1 is the salt concentration at the surface of the root and F is the solute flux into the root. However, we know that [7]

$$S_1 = \frac{1}{[1 + (-p_1)^n]^m}, \ and \ F = \lambda_w[p_1 - p_r(p_1)] \qquad (43)$$

So the equation (29) will be

$$p_1 = p_r - \frac{\partial^2 p_r}{k^2 \partial z^2} \qquad (44)$$

using these three relations (equation (43),(44)), we write the equation (42) in terms of pr will be

$$\frac{\partial^2 p_r}{\partial z^2}[1 + (\frac{\partial^2 p_r}{k^2 \partial z^2} - p_r)^n]^m \frac{\lambda_w L(t)}{2k^2 D(S_\infty)} + S_\infty[1 + (\frac{\partial^2 p_r}{k^2 \partial z^2} - p_r)^n]^m - 1 = 0 \qquad (45)$$

where . $L(T) = ln[4e^{-\gamma}D(S_\infty)t + 1]$

The above equations are solved for pr subject to boundary conditions

$$p_r = T_0 \ at \ z = 0 \ and \ \frac{\partial p_r}{\partial z} = 0 \ at \ z = 1 \qquad (46)$$

where $T_0 = \frac{T\alpha}{\rho g}$. For $\lambda_w \ll 1$ the leading order O(1) solution for pr is

$$p_r(z) = -(S_\infty^{-1/m} - 1)^{1/n}[1 + \tanh(k)\sinh(kz) - \cosh(kz)] - T[\tanh(k)\sinh(kz) - \cosh(kz)] + O(\lambda_w) \qquad (47)$$

The result for various different far-field values S_∞ are presented in Figure 2. So the salt uptake F(z) along the root is given by

$$F(z) = -\frac{\lambda_w}{k^2}\frac{\partial^2 p_r}{\partial z^2} + O(\lambda_w^2) \qquad (48)$$

and the total uptake of salt by this root systems given at the leading order by

$$F_{tot} = 2\pi\int_0^1 F(z)dz = -\frac{2\pi\lambda_w}{k^2}\frac{\partial p_r}{\partial z}\Big|_{z=0}^{z=1} = \frac{2\pi\lambda_w}{k^2}\frac{\partial p_r}{\partial z}\Big|_{z=0} \qquad (49)$$

As we can see the overall salt uptake by root is determined by the internal pressure gradient at the base of the root. Negative

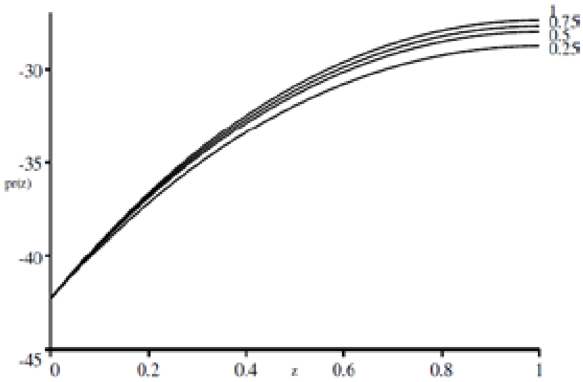

Fig 2: Vertical distribution of dimensionless pressure inside the root for various different far-field salt conc. given by equation (47). Other parameters T = -42.3, n = 2, m = 1- 1/n = 0.5

flux on Figure 3 implies that the root is bleeding out which happens at very low relative salt concentration.

4. Conclusion

In this paper we started by modelling the salt uptake by a cylindrical root in an infinite extent of soil. We have shown that the salt uptake by a single cylindrical root in the absence of competition does not influence very much the overall salt concentration in the soil even when the solute content in the soil is less than full saturation. Hence, we can now say that the Nye-Tinker-Barber model holds even for non-saturated soils provided that the dimensionless uptake parameter $|\lambda_w pr|$ is small and for all available experimental data on root tissue conductivities and agricultural soils $|\lambda_w pr|$ << 1. Hence we can see salt uptake by a plant root in infinite extent of soil does not influence the salt concentration profiles in saturated and unsaturated conditions.

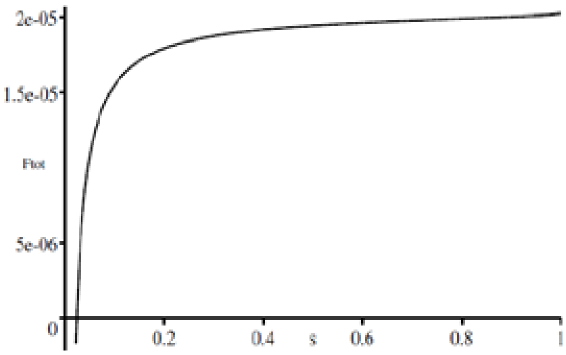

Fig 3. Total dimensionless salt uptake by the root as a function of far-field salt conc., given by equation (49). Other parameters T = -42.3, n =2, m =(1-1/n) = 0.5.

Acknowledgement: The work is financially supported by Ministry of Environment & Forests, Government of India (Project 19-20/2007-RE)

REFERENCE

1. Fahn. Plant Anatomy. Pergamon Press, 4 edition, 1990. | 2. J. J. Oertli. Transport of water in the rhizosphere and in roots., pages 559-587. Marcel Dekker Inc., New York, 2 edition, 1996. | 3. E. Steudle and C.A. Peterson. How does water get through roots? Journalof Experimental Botany, 49(322):775-788, 1998. | 4. F. J. Molz. Models of water transport in the soil-plant system. Water Resources Research, 17(5): 1245-1260, 1981. | 5. A. C. Fowler. Mathematical Models in the Applied Sciences. Cambridge University Press, 1997. | | 6. R. A. Freeze and J. A. Cherry. Groundwater. Prentice-Hall, Inc., 1979. | 7. M.Th.van Genuchten. A closed-form equation for predicting the hydraulic conductivity of unsaturated soils. Soil Science Society of America Journal, 44:892-898, 1980. | 8. Clausnitzer and J. W.Hopmans. Simultaneous modeling of transient threedimensional root growth and soil water o w. Plant and Soil, 164:299314, 1994. | 9. V. Clausnizer, J. W. Hopmans, and J. L. Starr. Parameter uncertainty analysis of common infiltration models. Soil Science Society of America Journal, 62:1477-1487, 1998. | 10. D-X. Chen and J. H. Lieth. Two dimensional model of water transport in the root zone and plant for container-grown crysanthemum. Agricultural and Forest Meterology, 59:129148, 1992. | 11. C. Doussan, L. Pages, and G. Vercambre. Modelling of the hydraulic architecture of root systems: An integrated approach to water absorption- model description. Annals of Botany, 81:213-223, 1998 | | 12. P. H. Nye. The effect of root shrinkage on soil water ino w. Philosophical Transactions of Royal Society of London B, 345:395-402, 1994. | 13. S. A. Barber. Soil Nutrient Bioavailability. A Mechanistic Approach. A Wiley-Inter science Publication, New York, 1984 |