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Approximate Derivative

Mathematics

KEYWORDS : - d(E, x)=approximate limit of E at x, where x belongs to the measurable set E,

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ABSTRACT

The functions one encounters in elementary analysis are usually differentiable or At least ,piecewise differentiable. Such functions are, for many purposes, sufficiently well behaved to give rise to smoothly flowing theories. One does not always need the full force of differentiability to obtain the results that one wants, however. Often, the existence of some sort of generalized derivative suffices for the development of at least part of the theory. For, example, the theorem which asserts that a function F, defined on an interval I, and having a positive derivative at each point of I, must be increasing, can be generalized in a number of ways through the use of generalized derivatives. The class of approximate derivatives is much larger than the class of derivatives. In this paper ,I can share my ideas about the behavior of approximate derivative and the study of structure of approximate derivative , which is the natural one to consider when dealing with the surface area.

Introduction

Of the many generalized derivatives which have been investigated, the one whose study fits most easily into the context of our first few chapters is the approximate derivative. Accordingly , we shall study this derivative in some detail in sections 1,2,3,below,Then ,in sections 4,we discuss a number of Miscellany briefly. Each of these approximate derivatives is obtained by a modification of the requirement that it be the limit of a difference quotient, and is therefore a “pointwise” derivative. There are also generalized derivatives which exist only in a “global “sense.

The Approximate Derivative – Basic properties.

We outline a development of the elementary properties of approximate derivatives.

Definition: Let F be measurable in a neighborhood of a point x0. The upper approximate limit of F at x0 is the greatest lower bound of the set { y: { x: F(x) > y } has x0 as a point of dispersion }. When the two are equal , their common value is called the approximate limit of F at x0.

We denote these limit by

Lim sup ap F(x) Lim inf ap F(x)

x→ x0 x→ x0

and Lim ap F(x) respectively.

x→ x0

It follows immediately from this definition that each approximate limit of F at x0 exists if and only if there exists a measurable set E such th$_{at}$

d(E, x0)=1 & lim F(x) = lim ap F(x)

x→ x0 x→ x0

for all x ε E

It is also clear that if two functions coincide on a measurable set, their approximate extreme limits agree at each point of density of that set.

Intuitively speaking , in dealing with approximate limits , we “ignore sets of density zero .”

The unilateral approximate extreme limits are defined in the obvious manner.

Now let F be any finite measurable function defined in a neighborhood of x0 and let

G(x, x0) = ,(x)

We define the approximate upper right derivate,

The approximate lower right derivative

The approximate upper left derivative

The approximate lower left derivative

The approximate bilateral upper derivate and

The approximate bilateral lower derivate as the corresponding approximate extreme limits of G(x, x0) as x→ x0 .

When all of these derivatives are equal , we call their common value the approximate derivative of F at x0 and denote it by).

When the approximate derivative of F at x0 is finite, we say F is approximately differentiable at x0 or F is approximately derivable at x0.

We state ,without proof, two theorems which indicate relationships between approximate differentiability and generalized bounded variations.

Theorem1: If F is measurable and generalized bounded variation on a set E, then F is approximately derivable a. e. on E.

Theorem 2: If at every point x of a measurable set E, a measurable function F satisfies at least one of the inequlities

Then F is generalized bound variation.

Thus, if a measurable function F satisfies at least one of the inequalities of Theorem 2. On a measurable set E, we see from Theorem 1, that F is approximately differentiable at almost every point of E.

Although we shall have no need for it in the sequel i.e. we do not assume F measurable the definitions of the approximate extreme derivates are the same as in the measurable case except that Lebesgue measure in the definition of concepts involving density.

Theorem 3: Let F be defined on an interval I, With the possible exception of a null set, I can be decomposed into four sets:

Set P, on which

F has a finite approximate derivative:

Set Q, on which

Set R, on which

=, , and

Set S, on which

=, =

If F is measurable , then the sets Q and R must also be null sets.

2.Behavior of approximate derivatives.

It is convenient to begin this section with an example. Let {In} be a sequence of disjoint closed intervals none of which contains the origin but whose union has the origin as point of density. Let F(0)=0, Fon In . Then, no matter how F is defined elsewhere, F will be approximately differentiable at x=0, and .It is easy to define F in such a manner as to be differentiable on all of

R {0}, but not differentiable at x = 0.

We can even construct F to be of bounded variation on R . Now let P be a nonempty nowhere dense perfect set contained in [0 1], To each interval Jn contiguous to P corresponds a subinterval [an b n] whose midpoint is the midpoint of Jn and whose length satisfies b n-an=J n)/n.

For each n, let Fn be a function defined on

[an b n] such that

F(an)=F(b n)=n)=n)=0,such that max{F(x): an<x<bn}=1 and such that F is differentiable on (an ,b n], and approximately differentiable (but not differentiable at an.

Finally ,define F to equal Fn on [an b n] and to vanish elsewhere. It is easy to verify that F is approximately differentiable, that on P {an}, and that F is differentiable every where except on that set. If we choose P to have positive measure, then we see that

an approximately differentiable function can fail to be differentiable function can fail to be differentiable on a set of positive measure

P {an}. note , however , that this set is nowhere dense. Note also that ,and that F is approximately continuous. It is not continuous, but we could easily modify F so as to be continuous and still exhibit the other properties we mentioned.

3.study of the structure of approximate derivatives.

We turn now to a study of the structure of approximate derivatives and of approximately differentiable functions. We shall see that although the class of approximate derivatives contains the class of derivatives properly, each approximate derivative possesses many of the desirable properties shared by all derivatives. We shall see that an approximate derivative P {an is always in BAIRE's first category.

Is always an actual derivative on a dense open set, and will be an actual derivative if it is bounded, or even if it is dominated by a derivative. We shall emphasize finite approximate derivatives, and then discuss briefly the complications that arise when we allow approximate derivatives to assume infinite values.

We shall make use of some of these simpler proof's in our development. In particular, we shall make use of some of the ideas we developed . we begin with a lemma.

Lemma 1: Let f be defined on some interval I. If f is the limit of a convergent interval function

Proof: Suppose f. We can find positive integers n and k, a nonempty perfect subset Q of I and a set Qk such that following are equivalent.

1. for all x Q
2. Qk is dense in Q
3.| Qk.

Theorem: Let F be approximately differentiable on I. Then the function f=.

Proof: By above lemma, it suffices to show that f is the limit of a convergent interval function .In order to define ,let us first define for each interval I and

each ά ε R a set A(I, ά) in by
A(I, ά)={(x,y): x,y I and > ά }

Since F is approximately differentiable, F is also approximately continuous and therefore measurable. Thus A(I, ά) is measurable subset of .Letting denote two dimensional Lebesgue measure, we now define

>1/2}

We show converges to f. Let x0 I0 and let {In} be a sequence of intervals in I0 such that

x0 In and λ(In)0. We must show lim(In) = f(x0).

Write ζ =f(x0) and let 1/8 >>0. There exists a positive integer N such that for n>N

< on a set E

for which .

Now fix nN, for x En=EIn, let

E(x)={y:yεEn and |x-x0|<8|x-y|}.

The set E(x) consists of those points in En which, for our purposes, are "not too close to x." using elementary facts about inequalities, we can readily verify that if x En and y E(x), then

<17,and if x En then λ(E(x))>λ(In)(1-) .

It now follows from Fubini's theorem that the set of points (x,y) In X In for which

<17 has measure at least

(In X In)(1-)2

Thus|In)-ζ|<17ε. It follows that lim (In) =f(x0) and the proof of the theorem is complete.

We shall now show that a monotonic function F is differentiable at each point of approximate differentiability and that if and that if F is monotonically increasing. It will then follow that each approximately differentiable function on some dense open set.

Lemma: Let F be an increasing function defined on an interval I0 For each x0 I0 ,

=. The corresponding equalities for the other extreme derivates and extreme approximate derivates are also valid.

Theorem: Let F be increasing on an interval I0.If F is approximately differentiable at x0 and F1(x0)=

Proof: If F is approximately differentiable at x0 , then the four approximate extreme unilateral derivates are equal at that point.

The theorem now follows immediately from lemma, that there are examples of continuous functions of bounded variations for which the analogue of theorem fails. so the functions we defined at the beginning of this section furnishes an example if it is suitable defined on the set where it does not vanish.

We state the following theorems ,with out proof, which are indicates the behavior of approximate derivatives in measure theory.

Theorem:

If F is approximately differentiable on an interval I0 and I0 ,then F is increasing on I0

Theorem: Let F be approximately differentiable on I0 and let G be differentiable on I0. If G1(x) for all xε I0,then F is differentiable on I0 and

F1(x) = for all x ε I0.

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Theorem: Let F be approximately differentiable on I0, Then there exists a dense open subset of I0 on which F is differentiable.

Theoem: Let F be approximately differentiable on I0 and let f = , then fε D

Corollary:(Mean value theorem). If F is approximately differentiable on I0 , then to each pair of numbers a,b ε I0 corresponds a number c between a and b such that (c)

Approximate Derivatives share another property with derivatives, the so-called Denjoy Property.

Theorem: Let F be approximately differentiable on I0.

If the set E={x : ά < <β} is not empty,then λ(E) >0.

4.Miscellany.

We discuss briefly certain additional topics related to approximate differentiably.

a. Infinite approximate derivative.

Without the hypothesis that is finite, the analysis becomes considerably more complicated.

If exists(finite or infinite),on I0,then it does not follow that FD, Even for FD, it is possible that F∉. The function F might fail to be approximately continuous on a set of cardinality c, though this set must be of the first category(and, of course, of measure 0)

b. Theories of Integration .

The usual definitions of the integrals of Lebesgue, Denjoy and Khintchine are all constructive in nature.The Lebesgue integral is defined by a limiting process involving certain sums, and the Denjoy and Khinchine integrals involve ceratin operations and some(transfinite) inductive processes.

Each of these integrals can also be defined in a so called "Descriptive" way that is, as an operation which is inverse to some already defined differentiation process.Thus ,we could give the following definitions(each of which would be a theorem if the constructive process were followed).

Definations: The function F is the Lebesgue integral of f on I0 provided

(1) F is Approximately Continuous on I0, and
(2) F1 = f a.e. on I0

The function F is the Denjoy integral of f on I0 provided

(1) F is Approximately continuously differentiable on I0 and
(2) F1= f a.e. on I0

The function F is the khintchine integral of f on I0 provided

(1) F is approximately continuously differentiable on I0
(2) =f a.e. on I0

c. Functions of several variables. We have been concerned almost entirely with functions of one real variable. Because approximate differentiation plays an important role in certain parts of the theory of functions of several real variables we take a moment to discuss ways in which approximate differentiation is "superior" to ordinary differentiation in this setting.

We have already observed ,that the Dini derivates of a function F of one variable inherit the measurability properties of F. For functions of several variables this is not so, however. If F is a function of two or more variables and is Lebesgue(Borel) measurable, then the extreme partial derivatives of F might fail to be Lebesgue measurable.(It is true, however, that if F is continuous(borel measurable),then these derivates will be Borel(resp.Lebesgue) measurable). The approximate extreme partial derivates do inherit the measurability properties of F, however.

Let F be continuous in some region D in R2. If the partial derivatives exist a.e. in D, must F be differentiable; But, if F has approximate partial derivatives a.e. in D, then F will be approximately differentiable a.e. in D. In this case, F will also have directional derivatives in almost every direction at almost every point.

Conclusion:

To establish the desired applications of approximate differentiability to explain the inner concept of monotonic increasing functions are approximately differentiable on some dense open set. In addition to the approximate derivative , a number of other generalized derivatives may be defined and studies by Mathematicians and Scientists. I am not state explicitly the obvious modifications necessary to define the unilateral or extreme derivates. It supports the Riemann, Taylor and Peano derivatives ,which are having important applications in approximation theory.

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