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Monotonicity

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ABSTRACT

Theorem in elementary calculus asserts that if F has a positive derivative at each point of some interval, then F is increasing on that interval. This theorem has been generalized in a number of ways. Many of the monotonicity theorems are very similar in nature and differ only in which generalized extreme derivates are considered. In this paper we discuss some of the generalization of monotonicity theorem's for increasing our understanding the concept of monotonicity.

Introduction:
In order to prepare for our development , it is convenient to state the elementary theorem in a very redundant manner.

Theorem: Let F be defined on an interval I0 .

If

(1) F is differentiable on I0,
(2) F1 exists everywhere on I0, and
(3) F1>0 everywhere on I0
Then F is increasing on I0.

We stated this theorem in our redundant form because it suggests three areas of generalization. We amplify this remark.

We view

condition (1):
As a regularity condition on F. An avenue of generalization is created by weakening this regularity condition, e.g., perhaps we wish to require only that F be approximately continuous, or F ε , or that F be in some Baire class.

Condition (2):
Can be generalized by replacing the ordinary derivative with some generalized derivative and /or by requiring only that the (perhaps generalized) derivative exist a. e., or on some residual set, or on the complement of some countable set.

Similarly ,

condition(3) :
Can be weakened by requiring only that the (perhaps generalized) derivative is nonnegative on some large set.

It is clear that the number of "conjectures" one can obtain by weakening conditions (1),(2), and /or (3) in all conceivable reasonable ways, is very large.

Some of these weakening result in theorems , and some do not. It is not our purpose here to either prove or disprove each conceivable statement. Instead , we shall outline some of the results that have proved to be important historically, then prove a general theorem which reduces certain cases to others, and then give indications of what some of the other known theorems are.

Before proceeding further, we note that the negative of the Cantor functions provides a simple counter example to a number of "reasonable' conjectures.

1.Some historical background :
The elementary theorem is a very simple one, but it often happens that the hypotheses of that theorem are not met, yet one wishes to prove that the function in monotonic. Or, sometimes the hypotheses are met, but one cannot prove this easily. It therefore is desirable to know more general conditions under which a functions must be monotonic.

We begin with a standard result in the theory of Lebesgue integration and proceed with a discussion of some of the more important ways in which it has been extended.

Theorem: If F is absolutely continuous on I0 and

F10 a.e. then F is no decreasing.

The negative of the Cantor function shows that one cannot replace absolute continuity with continuity in the enunciation of this theorem .However, some mathematicians showed in 1930's the following result holds if we only assume F to be continuous.

Theorem: Let F satisfy the following conditions on I0:
(i) F is continuous,
(ii) F1 exists (finite or infinite)except perhaps in a denume able set,
(iii) F1 0 a.e.
Then F is no decreasing in I_0

Again improved of this theorem on 1940's as follows.
(i) F is approximately continuous,
(ii) exists(finite or infinite) except perhaps on a denumerable set,
(iii) Then F is continuous and no decreasing on I_0

Now generalization of this theorem in 1950's as follows. :
Let F satisfy the following conditions on I0:
(i) F ε D,
(ii) F1 exists (finite or infinite), except, perhaps on a denumerable set,
(iii) F1 0 a.e.

Then F is continuous and no decreasing on I.
We note that generalization of this theorem is stronger than previous , on the other hand, it is weaker because the conditions involve the ordinary derivative in place of approximate derivative.

We would like a theorem which implies both theorems. That is must a Darboux function satisfying conditions (ii) and (iii) of improved theorem be no decreasing.

Now,
hidden in generalized theorem, F must be in 1 ,which is satisfies both the conditions of above theorems as follows

Theorem:
Let F satisfy the following conditions on I0.

(i) F ε D_1,
(ii) exists(finite or infinite) except, perhaps, on a denumerable set,
(iii) Then F is continuous and no decreasing on I0.

Actually , this last theorem, as well as certain other theorems follow from a general theorem which we discuss below.

A general theorem.
The theorem of above section might appear to be rather technical or specialized. In a sense, they are. But they, and others like them, can arise naturally. Perhaps one does not have enough information to show directly that a function F is continuous, or that some generalized derivative is nonnegative everywhere. It might actually turn out that F is differentiable and that F1 0 everywhere, but this might not be readily verifiable. If the generalized derivative is one of several of the ones we discussed, it will automatically turn out that F ε D1 .And for such an F we have a general theorem which asserts (roughly) that if the conditions on the generalized derivative suffice to guarantee monotonicity whenever F is continuous, then they suffice to guarantee monotonicity whenever F ε D1.

We shall prove this theorem in this section, and then indicate applications of it in next section.

We begin with several preliminary results which describe the behavior of functions in D1 , that meet certain regularizing conditions.

Recall that a function F is said to satisfy the Banach's condition if almost every value taken by F is taken at most a denumerable number of times, we shall denote the class of all such functions by J2.

We shall also use freely any facts about analytic sets. Each analytics set is measurable; each no denumerable analytic set contains a nonempty perfect set, the image under a Borel function of a Borel set is an analytic Set.

Theorem1: Suppose F ε D1J2 on I0. Then for almost every y ε F(I0), the set Ey= {x : F(x) = y} either consists of a single point or contains a pair of isolated neighbors(i.e. , there exist x1 ,x2 ε Ey such that x1,x2 are isolated in Ey and the interval (x1,x20 contains no point of Ey .

Proof: If F is constant on I0 , the result is obvious. If not, let Y1 = {y: Ey is no denumerable } and Y2={y: there exists x ε Ey such that F achieves a strict relative extremer at x}. Then λ(Y1) = 0 by hypothesis, and Y2 is at most denumerable .

Let y ε F(I0)1 U Y2).If Ey contains a pair of isolated neighbors. Suppose then that Ey us denumerable . We distinguish two cases.

Case 1: Ey isisolated. For each x0 ε Ey there exists a maximal interval I⊂I0 such that I⋂ Ey= {x0}. If one of the endpoints of some such interval lies in EY , we are done. If not, then both endpoints of some such interval lies in Ey.. Let {In]= { (an , bn)} be the sequence of such maximal intervals, and let P=I0 UIn. The set P is clearly perfect and P ⋂ Ey=Ø . Since Ey is an isolated set and y Y2 , F(an)-y and F(bn)-y are of opposite sign for every n. It follows that the sets {x:F(x) >y } and {x:F(x)<y} are dense in P. But then F|P has no point of continuity contradicting our assumption that F ε 1 .This completes the proof for case I.

Case II: The derived set contains points of Ey. If for some x0 ε Ey ⋂ there exists a neighborhood J such that J⋂ Ey ⋂ ={x0} we are done. For, in that case there exists an interval J1⊂J containing points of Ey all of which are isolated in Ey. On the interval J1, the argument of case I applies. On the other hand, suppose that every neighborhood of every point

x ε Ey ⋂ contains a point x1 x in Ey ⋂ ; then the set Ey ⋂ is dense –in- itself. But Ey is of type Gδ. ,because F ε 1, and is closed,

so Ey ⋂ is dense –in-itself and of type Gδ. . This implies that Ey ⋂ is no denumerable, and this contradicts the assumption that y ε F(I) Y1.The proof the theorem is now complete.

Therorem2: Let F ε D1J2 on I0 .For every pair of real numbers ά<β,
the set Eάβ {x: ά <F(x) <β} is either empty or contains an interval.

Proof: Suppose Eάβ is not empty. Let x0 ε Eάβ .
If I0 ⊂ Eάβ, there is nothing to prove. Thus , suppose

x1εI0 Eάβ ,say x1>x0 and F(x1) ά. Therorem1 applied to the interval [x0 x1], implies that there exist ά0>ά and x2 such that x0<x2<x1,x2 is an isolated point of Eά0, and F does not attain a strict relative extreme at x2 . Thus, there exists an interval I⊂ [x0 ,x1] such that F ά0 on I. If also F<β on I, nothing remains to be proved. Otherwise , arguing as above , we find a number β0 < β and an interval J⊂I such that F β0 on J. Thus , ά < ά0 F(x) β0 <β for all x ε J; i.e., J⊂ Eάβ, the proof is complete.

Result of theorem2:
It follows immediately that F ε Dβ1J2, then F maps its set of points of continuity onto a dense subset of its range. (This is a consequence of the fact that a Baire 1 function has a point of continuity in each interval l). Our generalization supports quasi-continuous functions , as we know that A function F is called quasi-continuous , for each x0 ε I there exist a sequence of intervals {In} converging to x0 such that F|({x0} U UIn) is continuous at x0. It follows that F ε Dβ1J2 is quasi –continuous.

Example: There exists a function F having a bounded derivative on [0,1] such that F is nowhere monotonic. proof: Let E⊂[0, 1] now we can define λ(E)=1 and

[0,1] E is dense in [0,1]. Then it has a derivative which is positive on the dense set E and 0 on the dense set

[0,1] E

Before stating next theorem , we note that F is VBG on I0 , then F ε J2 and above theorems also valid for such functions which are also in Dβ1. We also note that while a function F which is VB on a set A need not be so on

Ac(consider the characteristic function of the rationales), F will be VB on Ac provided F(I ⋂ A) is dense in

F(I ⋂ Ac) for each interval I. This s so because each approximating sum for the variation of F on Ac can be approximated arbitrarily well by sums which approximate the variation of F on A. More generally, if A is dense in B and F(A⋂I) is dense in F(B⋂I) for each interval I, then F will be VB in B provided F is VB on A. (This condition is equivalent to requiring the graph of F over A to be dense in the graph of F over B).

Theorem 3: Let F ε Dβ1 on I0 , then there exists sequence {In} of intervals whose union is dense in I0 and each of which F is continuous and of bounded variation.

Proof: Let H be the set of points of continuity of F The set H is a residual subset of I0 because

F ε β1 . since F is VBG on all of I0 , it is certainly VBG on H. Therefore we can write H= where F is VB on each Hn. Since H is residual, there exists an interval J⊂I0 and an integer N such that HN is dense in H⋂J. Now , since F is continuous on H ,F(I ⋂ HN) is dense in F(I ⋂ H) for each interval I⊂J. It follows from the remark preceding the statement of the theorem , that F is VB on J⋂H. But H is dense in J and , for each interval I, F(I⋂H) is dense in F(I) because of the remark following the proof of theorem. Therefore F is of bounded variation on all of J . It follows that any discontinuity of F in J must be a jump discontinuity on J.

The argument we have given applies equally well to any subinterval of I0. The conclusion of the theorem follows by repeated application of this proof.

Conclusion:
We note that we focused on conditions which imply monotonicity. We encountered nowhere monotonic functions, even ones

which are differentiable. Our theorems show that many functions in Dβ1 cannot satisfy BANACH'S conditions in T2 normal space. Also if hypothesis imply that the function F is VBG, and then follows under the assumption that F is continuous, F must be nondecreasing.

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