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Number of Subgroups of order 4 in S_n

Mathematics

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ABSTRACT

Permutation Group is an extra ordinary group of group theory. In this paper, the authors derive a general formula for number of subgroups of order 4 which may be cyclic or non-cyclic in permutation group of order n.

I. INTRODUCTION

A Permutation Group of n-element set has n! element, it has $2^{n!}-1$ non-empty subsets. This calculation is straightforward, but no easy calculation and no educated guess, to give the answer to following question:-

"How many out of** $2^{n!}-1$ **subsets are subgroup of** S_n **"

The objective of this paper is not answering this question, but part of the problem that ***"How many out of** $2^{n!}-1$ **subsets are subgroup of order 4 in** S_n **"***\

II. GENERAL CONCEPTS

If a subset H of a group G is itself a group under the operation of G, we say that H is a *subgroup* of G.

A *permutation* of a set A is a function from A to A that is both one-to-one and onto. A *permutation group* of a set A is a set of permutation of A that forms a group under function composition.

Order of an element is cardinality of set generated by that element.

III. USEFUL THEOREMS

Theorem 1: - Every permutation of a finite set can be written as a cycle or as a product of disjoint cycles.

Theorem 2:-If the pair of cycles $\alpha=(a_1,a_2,\dots\dots,a_m)$ and $\beta=(b_1,b_2,\dots\dots,b_n)$ have no entries in common, then $\alpha\beta=\beta\alpha$.

Theorem 3:-The order of a permutation of a finite set written in disjoint cycle form is the least common multiple of lengths of the cycles.

Theorem 4:- Every finite cycle group of order n has $\Phi(n)$ generators.

IV. CONSTRUCTION OF A CYCLIC SUBGROUP OF ORDER 4

From Theorem 1 & 3, we can conclude that every permutation of order 4 can be written as product of disjoint cycles of length 4 and 2 with at least one cycle of length 4. Each permutation of order 4 forms a cyclic subgroup. Now question arises how many elements are of order 4 are there in $\boldsymbol{S_n}$.

V. HOW MANY ELEMENTS OF ORDER 4 IN S_n

Here every permutation of order 4 from $\boldsymbol{S_n}$when decomposed into disjoint cycles have k 4-cycles, m 2-cycles and n-4k-2m 1-cycles where $1\le k\le\left[\frac{n}{4}\right]$ and $0\le m\le\left[\frac{n-4k}{2}\right]$.Therefore number of elements of order 4 are $\sum_{k=1}^{\left[\frac{n}{4}\right]}\sum_{m=0}^{\left[\frac{n-4k}{2}\right]}\frac{n!}{k!m!(n-4k-2m)!(4!)^k(2!)^m}$

VI. NUMBER OF CYCLIC SUBGROUP OF ORDER 4 IN S_n

From Theorem (3), we can conclude that every cyclic subgroup of order 4 has ϕ(4)=2 elements of order 4. Hence number of cyclic subgroup of order 4 in $S_n$are $\frac{1}{2}\sum_{k=1}^{\left[\frac{n}{4}\right]}\sum_{m=0}^{\left[\frac{n-4k}{2}\right]}\frac{n!}{k!m!(n-4k-2m)!(4!)^k(2!)^m}$

VII. CONSTRUCTION OF A NON-CYCLIC SUBGROUP OF ORDER 4

If α and β are two disjoint permutation of order 2, then $\{\alpha,\beta,\alpha\beta,e\}$ from a non-cyclic subgroup of order 4. If α and β are two not-disjoint permutation of order 2, then $\alpha\beta\neq\beta\alpha$. Therefore they cannot from a subgroup of order 4 because generated set of $\alpha\&\beta$ contains at least five elements$\alpha,\beta,\alpha\beta,\beta\alpha,e$. Now question arises how many disjoint pairs of order 2 are in S_n.

VIII. HOW MANY DISJOINT PAIRS OF ORDER 2 IN S_n

Here every permutation of order 2 from $S_n$when decomposed into disjoint cycles have m 2-cycles and n-2m 1-cycles where $0\leq m\leq\left[\frac{n}{2}\right]$.Therefore number of elements of order 2 are $\sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{n!}{m!(n-2m)!(2!)^m}$. But we needs number of disjoint pairs of order 2 in S_n.

If α contains m 2-cycles then numbers of elements out of which α can be selected are $\frac{n!}{m!(n-2m)!(2!)^m}$where $1\leq m\leq n-2$. Now we have to chosen βin such a way that $\alpha\&\beta$ are disjoint. Therefore we have to form of β out of n-2m elements such that order of it remains 2. Number of such $\beta's$are $\sum_{k=1}^{\left[\frac{n-2m}{2}\right]}\frac{(n-2m)!}{k!(n-2m-2k)!(2!)^k}$.

Therefore total number of disjoint pairs of order 2 in $S_n$are $\sum_{m=1}^{\left[\frac{n}{2}\right]}\left[\frac{n!}{m!(n-2m)!(2!)^m}\sum_{k=1}^{\left[\frac{n-2m}{2}\right]}\frac{(n-2m)!}{k!(n-2m-2k)!(2!)^k}\right]$

$=\sum_{m=1}^{\left[\frac{n}{2}\right]}\sum_{k=1}^{\left[\frac{n-2m}{2}\right]}\frac{n!}{k!m!(n-2k-2m)!(2!)^{m+k}}$

IX. GENERAL FORMULA FOR NUMBER OF SUBGROUP OF ORDER 4 IN S_n

We have just we have derived formula for number of subgroups of order 4 in both the cases (cyclic or non-cyclic). If add these two we get desired result.

Number of subgroup of order 4 in S_n are$\frac{1}{2}\sum_{k=1}^{\left[\frac{n}{4}\right]}\sum_{m=0}^{\left[\frac{n-4k}{2}\right]}\frac{n!}{k!m!(n-4k-2m)!(4!)^k(2!)^m}+$ $\sum_{m=1}^{\left[\frac{n}{2}\right]}\sum_{k=1}^{\left[\frac{n-2m}{2}\right]}\frac{n!}{k!m!(n-2k-2m)!(2!)^{m+k}}$

X. FUTURE WORK

One can find out number of subgroups in S_{14} of any order which is an open problem.

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