Some Rate Sequence Spaces

Mathematics

KEYWORDS : Rate sequence spaces, c_π , $(l_p)_\pi$, l_π^∞ , $(c_0)_\pi$, r_π , $\wedge_\pi$.

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INTRODUCTION:

In this paper, we study some rate sequence spaces and prove that they are complete metric spaces. A complex sequence, whose kth terms x_k is denoted by $\{x_k\}$ or simply x. Let Φ be the set of all finite sequence.

Definition.1

A metric d of X is said to be bounded if there exists a constant $K > 0$ such that $d(x, y) = \text{Sup}_{(n)}|x_n - y_n| < K$ where $x = (x_n)$, $y = (y_n) \in X$ and it is denoted by l_∞ .

Definition.2

A sequence $x = \{x_k\}$ in a metric space (X, d) is a Cauchy sequence if for every $\epsilon > 0$, there exists a positive integer n_0 such that

d(x,y) = $|x_n - x_m| < \epsilon \quad \forall\, m, n \geq n_0$

Let $\pi = \{\pi_k\}$ be a sequence of positive numbers. If X is a sequence space, then X_π denotes the rate space of X. In fact X_π is the vector space of all those sequences$\{x_k\}$ such that $\left\{\frac{x_k}{\pi_k}\right\} \epsilon X$.

Now

$$l_\pi^\infty = \left\{x = \left(\frac{x_k}{\pi_k}\right) : x \in l_\infty . \|x\| = \text{Sup}_{(k)} \left|\frac{x_k}{\pi_k}\right|\right\}$$

$$(c_0)_\pi = \left\{x = \left(\frac{x_k}{\pi_k}\right) : x \in c_0 . \|x\| = \text{Sup}_{(k)} \left|\frac{x_k}{\pi_k}\right|\right\}$$

$$c_\pi = \left\{x = \left(\frac{x_k}{\pi_k}\right) : x \in c. \quad \|x\| = \text{Sup}_{(k)} \left|\frac{x_k}{\pi_k}\right|\right\}$$

$(c_0)_\pi$ and c_π have the same metric as in l_π^∞.

$$(l_p)_\pi = \left\{x = \left(\frac{x_k}{\pi_k}\right) : x \in l_p . \ \|x\| = \left(\sum_{k=1}^{\infty} \left|\frac{x_k}{\pi_k}\right|^p\right)^{1/p} \ Hence\ d(x, y) = \|x - y\|\right\}$$

$$r_\pi = \left\{x = \left(\frac{x_k}{\pi_k}\right): x \in r.\ d(x,y) = \text{Sup}_{(k)} \left|\frac{x_k}{\pi_k} - \frac{y_k}{\pi_k}\right|^{1/k}\right\}$$

$$\wedge_\pi = \left\{x = \left(\frac{x_k}{\pi_k}\right): x \in \wedge.\quad d(x,y) = \text{Sup}_{(k)} \left|\frac{x_k}{\pi_k} - \frac{y_k}{\pi_k}\right|^{1/k}\right\}$$

Therem.1 l_π^∞ is a complete metric space.

Proof:

Let $\{x^{(n)}\}_{n=1}^\infty$ $with\ x^{(n)} = \left\{\frac{x_1{}^{(n)}}{\pi_1}, \frac{x_2{}^{(n)}}{\pi_2}, \ldots\right\}$ be a Cauchy sequence in l_π^∞. Then given $\epsilon > 0$ there exists a positive integer n_0 such that

$$\left\|\frac{x_m}{\pi_m} - \frac{x_n}{\pi_n}\right\| < \epsilon\ for\ all\ m,n \geq n_0$$

Hence $\left|\frac{x_k{}^{(m)}}{\pi_k} - \frac{x_k{}^{(n)}}{\pi_k}\right| < \epsilon$

$$for\ all\ m,n \geq n_0\ \ and\ \forall\ k. \quad \_\_\_\_\_\ (1)$$

Thus, for each k, the sequence $\{x_k{}^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence in c_π . But c_π is complete.

Hence there is a sequence $x = \left(\frac{x_k}{\pi_k}\right)$ such that $\frac{x_k{}^{(n)}}{\pi_k} \rightarrow \frac{x_k}{\pi_k}\ for\ all\ k\ as\ n \rightarrow \infty$.

We show that $x \in\ l_\pi^\infty$.

we know that Cauchy sequences are bounded. Hence, there exists a $K > 0$ such that

$$\left\|\frac{x^{(n)}}{\pi}\right\| \leq K\ \ \forall\ n$$

Therefore $\left|\frac{x_k{}^{(n)}}{\pi_k}\right| \leq K\ \ \forall\ n,k$

Consequently

$$\lim_{n\to\infty} \left|\frac{x_k{}^{(n)}}{\pi_k}\right| = \left|\frac{x_k}{\pi_k}\right| \leq K\ \ \forall\ k.$$

Or equivalently, $x \in l_\pi^\infty$. Letting $m \rightarrow \infty$ in (1), we obtain

$$\left|\frac{x_k}{\pi_k} - \frac{x_k{}^{(n)}}{\pi_k}\right| \leq \epsilon\ \forall\ n \geq n_0\ and\ \forall\ k,$$

Hence

$$\left\|\frac{x_k}{\pi_k} - \frac{x_k{}^{(n)}}{\pi_k}\right\| = \text{Sup}_{(k)} \left|\frac{x_k}{\pi_k} - \frac{x_k{}^{(n)}}{\pi_k}\right|$$

$$\leq \epsilon\ \ \forall\ n \geq n_0\ .$$

Thus, the arbitrary Cauchy sequence $\left(\frac{x^{(n)}}{\pi}\right)$ in l_π^∞ converges to an element $\frac{x}{\pi} \in\ l_\pi^\infty$.

Therefore l_π^∞ is a complete metric space.

Theorem.2 c_π is a closed subset in l_π^∞.

Proof:

Let $= \left(\frac{x_i}{\pi_i}\right) \in \bar{c}_\pi$, the closure of c_π .

Then there exists a sequence $\{x^{(n)}\}_{n=1}^\infty$ $with\quad x^{(n)} = \left\{\frac{x_1{}^{(n)}}{\pi_1}, \frac{x_2{}^{(n)}}{\pi_2}, \ldots\right\}$ such that

$$\frac{x^{(n)}}{\pi} \rightarrow \frac{x}{\pi}\ for\ all\ k\ as\ n \rightarrow \infty$$

Hence given $\epsilon > 0$, there exists a positive integer n_0 such that

$$\left|\frac{x_i{}^{(n)}}{\pi_i} - \frac{x_i}{\pi_i}\right| \leq \left\|\frac{x^{(n)}}{\pi} - \frac{x}{\pi}\right\| < \epsilon\ \forall\ i$$

and $\forall\ n \geq n_0$

In particular, $\frac{x^{(n_0)}}{\pi} \in c_\pi$ and

$$\left|\frac{x_i{}^{(n_0)}}{\pi_i} - \frac{x_i}{\pi_i}\right| < \frac{\epsilon}{3}\ \ \forall\ i$$

Volume : 2 | Issue : 10 | October 2013 • ISSN No 2277 - 8179

Since $\frac{x^{(n_0)}}{\pi} \in c_\pi \Rightarrow \left\{\frac{x_i{}^{(n_0)}}{\pi_i}\right\}_{i=1}^{\infty}$ is a Cauchy sequence.

$\Rightarrow \left|\frac{x_i{}^{(n_0)}}{\pi_i} - \frac{x_k{}^{(n_0)}}{\pi_k}\right| < \frac{\epsilon}{3} \quad \forall\, i, k \geq n_0$, for some n_0.

Hence $\left|\frac{x_i}{\pi_i} - \frac{x_k}{\pi_k}\right| \leq \left|\frac{x_i}{\pi_i} - \frac{x_i{}^{(n_0)}}{\pi_i}\right| + \left|\frac{x_i{}^{(n_0)}}{\pi_i} - \frac{x_k{}^{(n_0)}}{\pi_k}\right| + \left|\frac{x_k{}^{(n_0)}}{\pi_k} - \frac{x_k}{\pi_k}\right|$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

$for\ all\ i, k \geq n_0$

This show that $\frac{x}{\pi} = \left(\frac{x_i}{\pi_i}\right) \in c_\pi$.

Thus $\bar{c}_\pi = c_\pi$ in l_π^∞.

Corollary.1 c_π is a complete metric space.

Theorem.3 $(c_0)_\pi$ is closed in c_π.

Proof:

Let $= \left(\frac{x_i}{\pi_i}\right) \in (\bar{c}_0)_\pi$, the closure of $(c_0)_\pi$ in c_π

$\Rightarrow$ there exists a sequence $\left\{\frac{x^{(n)}}{\pi}\right\}$ in $(c_0)_\pi$ converging to x.

$\Rightarrow \left|\frac{x^{(n)}}{\pi} - \frac{x}{\pi}\right| < \frac{\epsilon}{2}\ for\ all\ n \geq n_0$ and for all $\epsilon > 0$ where n_0 is a positive integer.

$\Rightarrow \left|\frac{x_k{}^{(n)}}{\pi_k} - \frac{x_k}{\pi_k}\right| < \frac{\epsilon}{2}\ for\ all\ n \geq n_0$ and for all k

Now,

$$\left|\frac{x_k}{\pi_k}\right| \leq \left|\frac{x_k}{\pi_k} - \frac{x_k{}^{(n)}}{\pi_k}\right| + \left|\frac{x_k{}^{(n)}}{\pi_k}\right|$$

$$< \frac{\epsilon}{2} + \left|\frac{x_k{}^{(n)}}{\pi_k}\right|$$

Since $\left\{\frac{x^{(n)}}{\pi}\right\} \in (c_0)_\pi$, we have $\left|\frac{x_k{}^{(n)}}{\pi_k}\right| < \frac{\epsilon}{2}$ for all $k \geq K$ for some positive integer K.

Thus $\left|\frac{x_k}{\pi_k}\right| < \epsilon$. Hence $x \in (c_0)_\pi$, accordingly $(\bar{c}_0)_\pi = (c_0)_\pi$.

Corollary.2 $(c_0)_\pi$ is a complete metric space.

Theorem.4 Let $1 \leq p < \infty$, then $\left(l_p\right)_\pi$ is a complete metric space.

Proof: Let $\left\{x^{(n)}\right\}_{n=1}^{\infty}\ with\ x^{(n)} = \left\{\frac{x_1{}^{(n)}}{\pi_1}, \frac{x_2{}^{(n)}}{\pi_2}, \ldots\right\}$ be a Cauchy sequence in $\left(l_p\right)_\pi$. Then for every $\epsilon > 0$ there exists a positive integer n_0 such that

$$\left\|\frac{x^{(n)}}{\pi} - \frac{x^{(m)}}{\pi}\right\| < \epsilon\ for\ all\ m, n \geq n_0$$

and so $\sum_{k=1}^{\infty} \left|\frac{x_k{}^{(n)}}{\pi_k} - \frac{x_k{}^{(m)}}{\pi_k}\right|^p < \epsilon^p$ for all m,n$\geq n_0$

Therefore, for each fixed k,

$\left|\frac{x_k{}^{(n)}}{\pi_k} - \frac{x_k{}^{(m)}}{\pi_k}\right|^p < \epsilon^p$ for all m,n$\geq n_0$

Which gives

$\left|\frac{x_k{}^{(n)}}{\pi_k} - \frac{x_k{}^{(m)}}{\pi_k}\right| < \epsilon\ for\ all\ m, n\ \geq n_0$

$\Rightarrow \left\{\frac{x_k{}^{(n)}}{\pi_k}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in c_π.

But c_π is a complete.

Thus $\frac{x_k{}^{(n)}}{\pi_k} \rightarrow \frac{x_k}{\pi_k}\ \ as\ \ n \rightarrow \infty$.

Let $x = \left(\frac{x_k}{\pi_k}\right)$. We now show that $x \in \left(l_p\right)_\pi$. For any positive integer n_0, we have

$$\left\|\frac{x}{\pi}^{(m)} - \frac{x}{\pi}^{(n)}\right\|^p = \sum_{k=1}^{j}\left|\frac{x_k}{\pi_k}^{(m)} - \frac{x_k}{\pi_k}^{(n)}\right|^p$$

$$+ \sum_{k=j+1}^{\infty}\left|\frac{x_k}{\pi_k}^{(m)} - \frac{x_k}{\pi_k}^{(n)}\right|^p$$

$$< \epsilon^p \text{ for all m,n} \geq n_0$$

So that $\sum_{k=1}^{j}\left|\frac{x_k}{\pi_k}^{(n)} - \frac{x_k}{\pi_k}^{(m)}\right|^p < \epsilon^p$

Letting $n \to \infty$, we get

$\sum_{k=1}^{j}\left|\frac{x_k}{\pi_k} - \frac{x_k}{\pi_k}^{(m)}\right|^p \leq \epsilon^p$ ____(2)

Since this holds for all j, we conclude that

$\sum_{k=1}^{\infty}\left|\frac{x_k}{\pi_k} - \frac{x_k}{\pi_k}^{(m)}\right|^p \leq \epsilon^p$ ___ (3)

By Minkowski's Inequality, we obtain

$$\left(\sum_{k=1}^{j}\left|\frac{x_k}{\pi_k}\right|^p\right)^{1/p} \leq \left(\sum_{k=1}^{j}\left|\frac{x_k}{\pi_k} - \frac{x_k}{\pi_k}^{(m)}\right|^p\right)^{1/p}$$

$$+\left(\sum_{k=1}^{j}\left|\frac{x_k}{\pi_k}^{(m)}\right|^p\right)^{1/p} \quad \text{\_\_(4)}$$

$$\leq \epsilon^p + \left\|\frac{x}{\pi}^{(m)}\right\| \text{ using (2)}$$

But the Cauchy sequence $\left(\frac{x}{\pi}^{(m)}\right)$ is bounded and so there is a constant H > 0.

such that $\left\|\frac{x}{\pi}^{(m)}\right\| \leq H \ \forall m.$

using this in (4), we have

$\left(\sum_{k=1}^{j}\left|\frac{x_k}{\pi_k}\right|^p\right)^{1/p} \leq \epsilon^p + H.$

since j is arbitrary, it follows that

$\left(\sum_{k=1}^{\infty}\left|\frac{x_k}{\pi_k}\right|^p\right)^{1/p} \leq \epsilon^p + H.$

This shows that $x = \left(\frac{x_k}{\pi_k}\right)$ belongs to $\left(l_p\right)_\pi$.

Relation (3) yields

$\left\|\frac{x}{\pi} - \frac{x}{\pi}^{(m)}\right\| \to 0 \text{ as } m \to \infty$

Thus, the arbitrary Cauchy sequence $\left(\frac{x}{\pi}^{(n)}\right)$ in $\left(l_p\right)_\pi$ is converges.

Hence $\left(l_p\right)_\pi$ is complete.

Theorem.5 r_π is a complete metric space.

Proof:

Let $\{x^{(n)}\}_{n=1}^{\infty}$ $with\ x^{(n)} = \left\{\frac{x_1}{\pi_1}^{(n)}, \frac{x_2}{\pi_2}^{(n)}, \ldots\right\}$ be a Cauchy sequence in r_π . Then for every $\epsilon > 0$ there exists a positive integer n_0 such that

$\sup_{(k)}\left|\frac{x_k}{\pi_k}^{(n)} - \frac{x_k}{\pi_k}^{(m)}\right| < \frac{\epsilon}{2} \quad \forall\, m, n \geq n_0$

and so

$\left(\left|\frac{x_k}{\pi_k}^{(m)} - \frac{x_k}{\pi_k}^{(n)}\right|\right)^{1/k} < \frac{\epsilon}{2} \ \forall\, m, n \geq n_0 \ \forall k.$

$\Rightarrow \{x_k^{(m)}\}_{m=1}^{\infty}$ is a Cauchy sequence in c_π

$$\Rightarrow \frac{x_k}{\pi_k}^{(n)} \to \frac{x_k}{\pi_k},\ as\ n \to \infty$$

because c_π is a complete.

$$\left(\left|\frac{x_k}{\pi_k}^{(m)} - \frac{x_k}{\pi_k}\right|\right)^{1/k} < \frac{\epsilon}{2}\ for\ all\ k \quad \text{\_\_ (5)}$$

Fix m, there exists n_0 such that

$\left(\left|\frac{x_k}{\pi_k}^{(m)}\right|\right)^{1/k} < \frac{\epsilon}{2}\ for\ all\ n \geq n_0.$

Now

$$\left(\left|\frac{x_k}{\pi_k}\right|\right)^{1/k} \leq \left(\left|\frac{x_k}{\pi_k} - \frac{x_k}{\pi_k}^{(m)}\right|\right)^{1/k}$$

$$+\left(\left|\frac{x_k}{\pi_k}^{(m)}\right|\right)^{1/k}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\ for\ all\ n \geq n_0.$$

$\Rightarrow x = \left(\frac{x_k}{\pi_k}\right) \in r_\pi$. From (5) $\left\{\frac{x_k{}^{(m)}}{\pi_k}\right\}$ converges to x.

Hence r_π is a complete metric space.

Theorem.6 $\wedge_\pi$ is a complete metric space.

Proof:

Let $\{x^{(n)}\}_{n=1}^{\infty}$ $with\ x^{(n)} = \left\{\frac{x_1{}^{(n)}}{\pi_1}, \frac{x_2{}^{(n)}}{\pi_2}, \ldots\right\}$ be a Cauchy sequence in $\wedge_\pi$. Then for every $\epsilon > 0$ there exists a positive integer n_0 such that

$$\left|\frac{x^{(m)}}{\pi} - \frac{x^{(n)}}{\pi}\right| < \epsilon \quad \forall\ m, n \geq n_0$$

$$\Rightarrow \left\{\text{Sup}\left|\frac{x_k{}^{(m)}}{\pi_k} - \frac{x_k{}^{(n)}}{\pi_k}\right|\right\}^{1/k} < \epsilon\ \forall\, m,$$

$$\Rightarrow\ \text{Sup}\left|\frac{x_k{}^{(m)}}{\pi_k} - \frac{x_k{}^{(n)}}{\pi_k}\right| < \epsilon^k < \epsilon$$

$$\Rightarrow \left|\frac{x_k{}^{(m)}}{\pi_k} - \frac{x_k{}^{(n)}}{\pi_k}\right| < \epsilon\ \ \forall\ m, n \geq n_0 \text{ --(6)}$$

Let $\left\{\frac{x_k{}^{(m)}}{\pi_k}\right\}_{m=1}^{\infty}$ is a Cauchy sequence in c_π . because c_π is a complete, so $\left\{\frac{x_k{}^{(m)}}{\pi_k}\right\}_{m=1}^{\infty}$ converges.

Let $\lim_{m\to\infty} \left\{\frac{x_k{}^{(m)}}{\pi_k}\right\} = \frac{x}{\pi}$.
Taking $n \to \infty$ in (6)
we obtain $\left|\frac{x_k{}^{(n)}}{\pi_k} - \frac{x}{\pi}\right| < \epsilon$

for all $k \geq n_0$ _____(7)

Now $\left|\frac{x}{\pi}\right| \leq \left|\frac{x}{\pi} - \frac{x_k{}^{(n)}}{\pi_k}\right| + \left|\frac{x_k{}^{(n)}}{\pi_k}\right|$

$$\leq \epsilon + k < \infty$$

$\Rightarrow\ x = \frac{x}{\pi} \in \wedge_\pi$. Thus $\wedge_\pi$ is a complete metric space.

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