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Some Special Types of Secondary Queues with Truncation of Probability

Statistics

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ABSTRACT

The present literature on queueing theory focuses on only one queue,which is formed in front of a server and does not focus on the queues formed when the server has no service to offer to the customers after some time. Moreover the services are such that they can be surrendered to the server by the customers after using it for some finite period of time. Here queueing systems have two different queues namely 'primary queue' and 'secondary queue'. The existence of secondary queue is possible only when server has limited service and the system has a characteristic of 'Service surrender facility'. This paper contains analysis of secondary queues where a model with finite range of service holding time and truncation of probability of service surrender to the right is considered. The expected waiting time of a customer in the entire queueing system and the rate of service wastage are obtained.

1. INTRODUCTION

A queueing system is composed of customers arriving for service, waiting for service if it is not immediate, and if having waited for service, leaving the system after being served or sometimes without being served. Up till now in the literature of queueing analysis, right from A.K. Erlang [2] to Feller,W [3], D.G. Kendall [5],D.V.Lindley [9], N.U. Prabhu and U.N. Bhat [10] and till the recent years the entire study is focused on the one and only one queue which is formed in front of a service counter.

S.P.Kane and P.B.Lakhani [4] pointed out that there are some queueing systems where a customer not only has to wait in single queue but also has to wait in another if did not served in the first queue due to limited service with the service counter. The existence of second queue is possible only when system has a characteristic of 'service surrender facility' which means returning back the utilized service by the customer.

It should be noted that in every queueing system it is not possible for a customer to surrender the service. Some examples where service surrender facility is not available are

1) A patient in a queue for taking service from a doctor.
2) A customer in a queue for food from a restaurant.
3) A customer being served by a barber.

In contrary, some examples where service surrender facility is available are

1) A queue of customers for acquiring a railway reservation ticket.
2) A queue of customers for getting a locker in the bank.
3) A queue of persons to borrow book(s) from the library.

1.1 Primary and Secondary Queues

Let there be a service counter where the service is available only to a finite number of customers (say N).Initially the system works and after offering the service to N customers, service is not available to any of the next waiting customers in queue. From this point of time waiting customers have an option either to register themselves into a 'waiting list' or to quit the system. This waiting list now becomes a secondary queue. Thus the customer initially joins the 'primary queue' and after reaching the service counter and knowing that all the services are exhausted has a choice whether to join the 'secondary queue' or to quit the system. The duration for maximum waiting time in the secondary queue is uncertain. Following the terminology of Bartholomew D.J.and Forbes A.F.[1], we define Service Holding Time (SHT) of the queueing system as the average duration of time for which a customer holds the service, before it is surrendered.

2. ANALYSIS OF THE SECONDARY QUEUE

Let p be the probability that a customer enjoys the service for a unit time. The probability that he surrenders the service some time during a unit time is (1-p) = q (say).Therefore the probability that a customer completes 't' units of time before he surrenders the service is a Geometric variable with p.m.f.

$P(T = t) = p^{t-1}(1-p)$ t = 1,2, . . ., ¥.

As the probability 'p' differs from person to person randomly, taking any value in the range [0,1], it is appropriate to consider it as a random variable.

The appropriate distribution of p may be considered as Beta distribution with parameters 'a' and 'b' which is given by

$f(p) = \frac{1}{B(a\ b)} p^{a-1}(1-p)^{b-1}$ a,b >0 ;0<p<1.

Hence the situation can be well studied by considering Compound Geometric Beta distribution. Thus SHT(T) can be assumed to follow a compound Geometric Beta distribution, which can be utilized to further analyze the secondary queues.

P.B.Lakhani and S.P.Kane [6] have given analysis of secondary queues where a model with infinite range of service holding time and truncation of probability of service surrender to the right i.e. $0<p\le\alpha_2$ where $0<\alpha_2<1$ is considered.

Here the distribution of p truncated to the right at a_2 is given by

$$f(p) = \begin{cases} K \frac{1}{B(a,b)} p^{a-1}(1-p)^{b-1} & 0 < p \le \alpha_2 \\ 0 & o.w. \end{cases}$$

The following probability function of compound Geometric right truncated Beta distribution is derived in this paper of P.B.Lakhani and S.P.Kane [6].

$$P(T = t) = f_t = \frac{B_{\alpha_2}(a+t-1, b+1)}{B_{\alpha_2}(a, b)} \quad a,b > 0; 0 < \alpha_2 < 1; t = 1,2 \ldots$$

1) To make the expression (1) more convenient to handle the values of a and b both are substituted equal to 1, in the above paper [6] and the following probability function is obtained.

$$\therefore P(T = t) = f_t = \frac{B_{\alpha_2}(t, 2)}{B_{\alpha_2}(1, 1)} \quad ...(2)$$

$$= \frac{\alpha_2^{t-1}[1+t(1-\alpha_2)]}{t(t+1)}$$

$0<a_2<1;t=1,2,.. ...(3)$

2.1 Model with finite range of SHT and 0 < p £ a_2 :
Let us consider a queueing system that works for some finite time and after that the entire system vanishes. This means that the customers cannot utilize the service for indefinite period of time. The service utilization stops after a fixed time period even after the customer does not surrender his service. Further this implies that if a customer wants, he can surrender his service before reaching that fixed time point and as such the next one in the queue gets the service. The customers who are on 'waiting list' are also in the system for a fixed time irrespective of the fact that they get the service or not.

So here SHT ranges from zero to some finite number say 'A'. Hence for the analysis of this situation we need to obtain the p.m.f. of the right truncated compound Geometric Beta distribution. Before that we make a few assumptions as follows :

Assumption 1 :
The event of surrendering the service and getting a service to a customer who is registered on the 'waiting list' are allowed to occur only at discrete time points.

Assumption 2 :
These time points are equipped spaced.

Let T be the SHT of a customer in the system . Let 'A' be the maximum length of service utilization.So here SHT ranges from zero to some finite number say 'A'. In other words we truncate T to the right at A and the range of 'p' is truncated to the right at a_2 . We consider this model for particular values of a and b (i.e. with a =1 and b =1).\From equation (3) we write p.m.f. of right truncated compound Geometric Beta distribution as

$$P(T=t)=f_t=\begin{cases} K.\frac{\alpha_2^{t-1}[1+t(1-\alpha_2)]}{t(t+1)} & 0<t\le A \\ 0 & \text{o.w.}\end{cases}$$

$$\sum_{t=1}^{\infty} f_t = 1 \quad \Rightarrow \sum_{t=1}^{A} f_t + \sum_{t=A-1}^{\infty} f_t = 1$$

$$\text{i.e.} \sum_{t=1}^{A} f_t = \sum_{t=1}^{A} K.\frac{\alpha_2^{t-1}[1+t(1-\alpha_2)]}{t(t+1)} = 1$$

$$\Rightarrow K = \frac{t(t+1)}{\sum_{t=1}^{A}\alpha_2^{t-1}[1+t(1-\alpha_2)]}$$

$$\therefore P(T=t) = f_t = \frac{\alpha_2^{t-1}[1+t(1-\alpha_2)]}{\sum_{t=1}^{A}\alpha_2^{t-1}[1+t(1-\alpha_2)]}$$

...(4)

$$\text{Consider} \sum_{t=1}^{A}\alpha_2^{t-1}[1+t(1-\alpha_2)]$$

$$= 2(1+\alpha_2+\alpha_2^2+\alpha_2^3+...+\alpha_2^{A-1}) - A\alpha_2^A$$

$$= \frac{2(1-\alpha_2^A)}{(1-\alpha_2)} - A\alpha_2^A$$

$$= \frac{2-(A+2)\alpha_2^A + A\alpha_2^{A+1}}{(1-\alpha_2)}$$

$$(4) \Rightarrow f_t = \frac{\alpha_2^{t-1}(1-\alpha_2)[1+t(1-\alpha_2)]}{[2-(A+2)\alpha_2^A + A\alpha_2^{A+1}]}$$

...(5)

2.1.1Average waiting time of a customer in the secondary queue

$$E(t) = \sum_{t=1}^{A} t\, f_t$$

Using (5) we get

$$E(t) = \sum_{t=1}^{A} t.\frac{\alpha_2^{t-1}(1-\alpha_2)[1+t(1-\alpha_2)]}{[2-(A+2)\alpha_2^A + A\alpha_2^{A+1}]}$$

$$= \frac{(1-\alpha_2)}{[2-(A+2)\alpha_2^A + A\alpha_2^{A-1}]}\left\{\begin{array}{l} 2+5\alpha_2+8\alpha_2^2+11\alpha_2^3 \\ +14\alpha_2^4 \\ +...+[3(A-1)-1]\alpha_2^{A-2} \\ +(3A-1)\alpha_2^{A-1} - A^2\alpha_2^A \end{array}\right\}$$

... (6)

Consider

$$S = 2 + 5\alpha_2 + 8\alpha_2^2 + 11\alpha_2^3 + ... + [3(A-1)-1]\alpha_2^{A-2} + (3A-1)\alpha_2^{A-1}$$

$$\alpha_2 S = 2\alpha_2 + 5\alpha_2^2 + 8\alpha_2^3 + ... + (3A-4)\alpha_2^{A-1} + (3A-1)\alpha_2^A$$

$$(1-\alpha_2)S = 2 + 3\alpha_2 + 3\alpha_2^2 + 3\alpha_2^3 + ... + 3\alpha_2^{A-1} - (3A-1)\alpha_2^A$$

$$= 2 + 3\alpha_2(1+\alpha_2+\alpha_2^2+...+\alpha_2^{A-2}) - (3A-1)\alpha_2^A$$

$$S = \frac{2}{(1-\alpha_2)} + \frac{3\alpha_2(1-\alpha_2^{A-1})}{(1-\alpha_2)^2} - \frac{(3A-1)\alpha_2^A}{(1-\alpha_2)}$$

$$= \frac{2(1-\alpha_2) + 3\alpha_2(1-\alpha_2^{A-1}) - (3A-1)(1-\alpha_2)\alpha_2^A}{(1-\alpha_2)^2}$$

$$(6) \Rightarrow E(t) = \frac{\left[\begin{array}{l}(2+\alpha_2) - (A^2+3A+2)\alpha_2^A + (2A^2+3A-1) \\ \alpha_2^{A+1} - A^2\alpha_2^{A+2}\end{array}\right]}{(1-\alpha_2)[2-(A+2)\alpha_2^A + A\alpha_2^{A+1}]}$$

...(7)

2.1.2The distribution function of T
The p.m.f. of right truncated compound Geometric Beta distribution is given by equation (5).Using this equation the distribution function is obtained as follows.

$$F_T = \sum_{t=1}^{T} f_t = \sum_{t=1}^{T} \frac{\alpha_2^{t-1}(1-\alpha_2)[1+t(1-\alpha_2)]}{[2-(A+2)\alpha_2^A + A\alpha_2^{A+1}]}$$

$$= \frac{(1-\alpha_2)}{[2-(A+2)\alpha_2^A + A\alpha_2^{A-1}]}\sum_{t=1}^{T}\alpha_2^{t-1}[1+t(1-\alpha_2)]$$

$$= \frac{(1-\alpha_2)}{[2-(A+2)\alpha_2^{A}+A\alpha_2^{A+1}]}\left\{2\left(\frac{1-\alpha_2^{T}}{1-\alpha_2}\right)-T\alpha_2^{T}\right\}$$

$$= \frac{[2-(T+2)\alpha_2^{T}+T\alpha_2^{T+1}]}{[2-(A+2)\alpha_2^{A}+A\alpha_2^{A+1}]}$$

…(8)

2.1.3 Survival function of the customer
The survival function of the customers in the system is

$$G_T = 1 - F_{(T-1)}$$

$$= 1 - \frac{[2-(T+1)\alpha_2^{T-1}+(T-1)\alpha_2^{T}]}{[2-(A+2)\alpha_2^{A}+A\alpha_2^{A+1}]}$$

$$= \frac{[A\alpha_2^{A+1}-(A+2)\alpha_2^{A}+(T+1)\alpha_2^{T-1}-(T-1)\alpha_2^{T}]}{[2-(A+2)\alpha_2^{A}+A\alpha_2^{A+1}]}$$

…(9)

2.1.4 Service wastage of the model
The rate of the service wastage of the model becomes

$$S_T = \frac{f_T}{G_T}$$

$$= \frac{\alpha_2^{T-1}(1-\alpha_2)[1+T(1-\alpha_2)]}{[A\alpha_2^{A+1}-(A+2)\alpha_2^{A}+(T+1)\alpha_2^{T-1}-(T-1)\alpha_2^{T}]}$$

…(10)

From (10) note that service wastage S_T is a function of T , α_2 and A. Here we developed a model where the range of 'p' is truncated to the right at α_2 i.e.$0<p\leq \alpha_2$ for particular values of parameters a and b with a=1 and b=1with finite range of SHT.For particular value of A,knowing the values of T and α_2, S_T gives the amount of instability in the system which is shown in the table-1 below. The various values of S_T presented in the table exhibits a monotonous behavior for fixed A and different sets of values of T and α_2.It is to note that for particular value of A and T<A, as α_2 goes on increasing, the service wastage S_T decreases in the beginning and then gradually stabilizes after some time. Further, to note that S_T will increase after a long time. This phenomenon is much closer to the reality of the model as in system the surrender pattern of customers will have a sharp decrease at the initial period because of various environmental and psychological factors and stabilizes when time passes and then increases due to deaths of the customers , new modern alternative and better service facilities etc.over long period. At the point T=A, value of service wastage S_T will be 1.

Table - 1
Values of S_T with finite range of SHT and $0<p\leq \alpha_2$

T	S_T				
	A=50				
	α_2 =0.25	α_2 =0.35	α_2 =0.5	α_2 =0.65	α_2 =0.75
5	0.712500	0.600543	0.437500	0.283088	0.187502
10	0.728571	0.621019	0.461538	0.305825	0.205888
15	0.735000	0.629505	0.472222	0.317029	0.215927
20	0.738462	0.634146	0.478261	0.323701	0.222286
25	0.740625	0.637074	0.482143	0.328133	0.226795
30	0.742105	0.639089	0.484849	0.331338	0.230589
35	0.743182	0.640560	0.486853	0.334109	0.235401
40	0.744000	0.641690	0.488672	0.339161	0.246912
45	0.744848	0.643900	0.498343	0.368032	0.293925
50	1.000000	1.000000	1.000000	1.000000	1.000000

T	S_T				
	α_2 =0.5				
	A=5	A=7	A=10	A=15	A=20
1	0.403361	0.383234	0.376286	0.375054	0.375002
2	0.450704	0.414239	0.402200	0.400093	0.400004
3	0.512821	0.441989	0.420499	0.416828	0.416673
4	0.631579	0.475248	0.435374	0.428856	0.428583
5	1.000000	0.528302	0.449799	0.438008	0.437520
6		0.640000	0.467153	0.445363	0.444481
7		1.000000	0.493151	0.451676	0.450066
8			0.540541	0.457633	0.454667
9			0.647059	0.464074	0.458557
10			1.000000	0.472325	0.461955
11				0.484848	0.465064
12				0.506787	0.468130
13				0.550459	0.471513
14				0.653061	0.475836
15				1.000000	0.482270
16					0.493151
17					0.513514
18					0.555556
19					0.656250
20					1.000000

P.B.Lakhani and S.P.Kane [7] have given analysis of secondary queues where a model with infinite range of service holding time and incomplete range (truncation at both the ends)of probability of service surrender is considered.

P.B.Lakhani and N.S.Kane [8] have given analysis of secondary queues where a model with finite range of service holding time and complete range of probability (i. e. from 0 to 1) of service surrender is considered.

3 GENERAL INTERPRETATIONS
Analysis of service surrender queues is dominated mainly by a secondary queue. The main focus is on finding the expected waiting time of a customer in the entire system.

Note that the customer's waiting time in the system can be divided into two parts. The one is waiting time in the primary queue and the second is waiting time in the secondary queue. Let the average waiting time in primary queue be W, depending upon the model which the primary queue follows.

The average waiting time in the secondary queue is given by equation (7). Therefore the overall average waiting time in the system could be expressed as

$$W + \frac{\begin{bmatrix}(2+\alpha_2)-(A^2+3A+2)\alpha_2^{A}\\ +(2A^2+3A-1)\alpha_2^{A+1}\\ -A^2\alpha_2^{A+2}\end{bmatrix}}{(1-\alpha_2)[2-(A+2)\alpha_2^{A}+A\alpha_2^{A+1}]}$$

The value of service wastage S_T is one of the most important characteristic of the model. It is a measure of disturbances in the secondary queue indicating the rate of service surrender. More the value of S_T more are the chances of getting the surrendered service to the new customers.

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