

Interval Valued Fuzzy Subrings of A Ring Under Homomorphism



Mathematics

KEYWORDS : Interval valued fuzzy subset, interval valued fuzzy subring, interval valued anti-fuzzy subring, pseudo interval valued fuzzy coset.

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ABSTRACT

In this paper, we study some of the properties of interval valued fuzzy subring of a ring under homomorphism and anti-homomorphism and prove some results on these. 2000 AMS SUBJECT CLASSIFICATION: 03F55,

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INTRODUCTION: Interval-valued fuzzy sets were introduced independently by Zadeh [12], Grattan-Guinness [5], Jahn [7], in the seventies, in the same year. An interval valued fuzzy set (IVF) is defined by an interval-valued membership function. Jun.Y.B and Kin.K.H[8] defined an interval valued fuzzy R-subgroups of nearrings. Solairaju.A and Nagarajan.R[10] defined the characterization of interval valued Anti fuzzy Left h-ideals over Hemirings. Azriel Rosenfeld[3] defined a fuzzy groups. Asok Kumer Ray[2] defined a product of fuzzy subgroups. We introduce the concept of interval valued fuzzy subring of a ring under homomorphism and anti-homomorphism and established some results.

1.PRELIMINARIES:

1.1 Definition: Let X be any nonempty set. A mapping $[M] : X \rightarrow D[0, 1]$ is called an interval valued fuzzy subset (briefly, IVFS) of X , where $D[0,1]$ denotes the family of all closed subintervals of $[0,1]$ and $[M](x) = [M^-(x), M^+(x)]$, for all x in X , where M^- and M^+ are fuzzy subsets of X such that $M^-(x) \leq M^+(x)$, for all x in X . Thus $M^-(x)$ is an interval (a closed subset of $[0,1]$) and not a number from the interval $[0,1]$ as in the case of fuzzy

subset. Note that $[0] = [0, 0]$ and $[1] = [1, 1]$.

1.2 Remark: Let D^X be the set of all interval valued fuzzy subset of X , where D means $D[0, 1]$.

1.3 Definition: Let $[M] = \{ \langle x, [M^-(x), M^+(x)] \rangle / x \in X \}$, $[N] = \{ \langle x, [N^-(x), N^+(x)] \rangle / x \in X \}$ be any two interval valued fuzzy subsets of X . We define the following relations and operations:

- (i) $[M] \subseteq [N]$ if and only if $M^-(x) \leq N^-(x)$ and $M^+(x) \leq N^+(x)$, for all x in X .
- (ii) $[M] = [N]$ if and only if $M^-(x) = N^-(x)$ and $M^+(x) = N^+(x)$, for all x in X .
- (iii) $[M] \cap [N] = \{ \langle x, [\min \{ M^-(x), N^-(x) \}, \min \{ M^+(x), N^+(x) \}] \rangle / x \in X \}$.
- (iv) $[M] \cup [N] = \{ \langle x, [\max \{ M^-(x), N^-(x) \}, \max \{ M^+(x), N^+(x) \}] \rangle / x \in X \}$.
- (v) $[M]^c = [1] - [M] = \{ \langle x, [1 - M^+(x), 1 - M^-(x)] \rangle / x \in X \}$.

1.4 Definition: Let $(R, +, \cdot)$ be a ring. An interval valued fuzzy subset $[M]$ of R is said to be an **interval valued fuzzy subring (IVFSR)** of R if the following conditions are satisfied:

- (i) $[M](x-y) \geq \min \{ [M](x), [M](y) \}$,
 - (ii) $[M](xy) \geq \min \{ [M](x), [M](y) \}$,
- for all x and y in R .

1.5 Definition: Let $(R, +, \cdot)$ be a ring. An interval valued fuzzy subset $[M]$ of R is said to be an **interval valued anti-**

fuzzy subring of R if the following conditions are satisfied:

- (i) $[M](x-y) \leq \max \{ [M](x), [M](y) \}$,
 - (ii) $[M](xy) \leq \max \{ [M](x), [M](y) \}$,
- for all x and y in R .

1.6 Definition: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two rings. Let $f : R \rightarrow R^1$ be any function and $[M]$ be an interval valued fuzzy subring in R , $[V]$ be an interval valued fuzzy subring in $f(R) = R^1$, defined by $[V](y) = \sup_{x \in f^{-1}(y)} [M](x)$,

for all x in R and y in R^1 . Then $[M]$ is called a pre-image of $[V]$ under f and is denoted by $f^{-1}([V])$.

1.7 Definition: Let $[M]$ be an interval valued fuzzy subring of a ring R . Then $[M]^0$ is defined as $[M]^0(x) = [M](x) / [M](0)$, for all x in R , where 0 is the identity element of R .

1.8 Definition: Let $[M]$ be an interval valued fuzzy subring of a ring $(R, +, \cdot)$ and a in R . Then the **pseudo interval valued fuzzy coset** $(a[M])^p$ is defined by $((a[M])^p)(x) = p(a)[M](x)$, for every x in R and for some p in P .

1.9 Definition: Let $[M]$ be an interval valued fuzzy subring of a ring $(R, +, \cdot)$. For any a in R , $a+[M]$ defined by $(a+[M])(x) = [M](x-a)$, for every x in R , is called an **interval valued fuzzy coset** of R .

2. SOME PROPERTIES:

2.1 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two rings. The homomorphic image of an interval valued fuzzy subring of R is an interval valued fuzzy subring of R^1 .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two rings and $f: R \rightarrow R^1$ be a homomorphism. That is $f(x+y) = f(x)+f(y)$, $f(xy) = f(x)f(y)$, for all x and y in R . Let $[M]$ be an interval valued fuzzy subring of R . Let $[V]$ be the homomorphic image of $[M]$ under f . We have to prove that $[V]$ is an interval

valued fuzzy subring of $f(R) = R^1$. Now, for $f(x)$ and $f(y)$ in R^1 , we have $[V](f(x) - f(y)) = [V](f(x-y)) \geq [M](x-y) \geq \min \{ [M](x), [M](y) \}$, which implies that $[V](f(x)-f(y)) \geq \min \{ [V](f(x)), [V](f(y)) \}$. And, $[V](f(x)f(y)) = [V](f(xy)) \geq [M](xy) \geq \min \{ [M](x), [M](y) \}$, which implies that $[V](f(x)f(y)) \geq \min \{ [V](f(x)), [V](f(y)) \}$. Hence $[V]$ is an interval valued fuzzy subring of a ring R^1 .

2.2 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two rings. The homomorphic pre-image of an interval valued fuzzy subring of R^1 is an interval valued fuzzy subring of R .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two rings and $f: R \rightarrow R^1$ be a homomorphism. That is $f(x+y) = f(x)+f(y)$, $f(xy) = f(x)f(y)$, for all x and y in R . Let $[V]$ be an interval valued fuzzy subring of $f(R) = R^1$. Let $[M]$ be the pre-image of $[V]$ under f . We have to prove that $[M]$ is an interval valued fuzzy subring of R . Let x and y in R . Then, $[M](x-y) = [V](f(x-y)) = [V](f(x) - f(y)) \geq \min \{ [V](f(x)), [V](f(y)) \} = \min \{ [M](x), [M](y) \}$, which implies that $[M](x-y) \geq \min \{ [M](x), [M](y) \}$, for x and y in R . And, $[M](xy) = [V](f(xy)) = [V](f(x)f(y)) \geq \min \{ [V](f(x)), [V](f(y)) \} = \min \{ [M](x), [M](y) \}$, which implies that $[M](xy) \geq \min \{ [M](x), [M](y) \}$, for x and y in R . Hence $[M]$ is an interval valued fuzzy subring of the ring R .

2.3 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two rings. The anti-homomorphic image of an interval valued fuzzy subring of R is an interval valued fuzzy subring of R^1 .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two rings and $f : R \rightarrow R^1$ be a anti-homomorphism. That is $f(x+y) = f(y)+f(x)$, $f(xy) = f(y)f(x)$, for all x and y in R . Let $[M]$ be an interval valued fuzzy

subring of R . Let $[V]$ be the homomorphic image of $[M]$ under f . We have to prove that $[V]$ is an interval valued fuzzy subring of $f(R) = R^1$. Now, let $f(x)$ and $f(y)$ in R^1 , we have $[V](f(x)-f(y)) = [V](f(y-x)) \geq [M](y-x) \geq \min \{ [M](x), [M](y) \}$, which implies that $[V](f(x)-f(y)) \geq \min \{ [V](f(x)), [V](f(y)) \}$. And, $[V](f(x)f(y)) = [V](f(yx)) \geq [M](yx) \geq \min \{ [M](x), [M](y) \}$, which implies that $[V](f(x)f(y)) \geq \min \{ [V](f(x)), [V](f(y)) \}$. Hence $[V]$ is an interval valued fuzzy subring of R^1 .

2.4 Theorem: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two rings. The anti-homomorphic pre-image of an interval valued fuzzy subring of R^1 is an interval valued fuzzy subring of R .

Proof: Let $(R, +, \cdot)$ and $(R^1, +, \cdot)$ be any two rings and $f : R \rightarrow R^1$ be a anti-homomorphism. That is $f(x + y) = f(y) + f(x)$, $f(xy) = f(y)f(x)$, for all x and y in R . Let $[V]$ be an interval valued fuzzy subring of $f(R) = R^1$. Let $[M]$ be the pre-image of $[V]$ under f . We have to prove that $[M]$ is an interval valued fuzzy subring of R . Let x and y in R . Now, $[M](x-y) = [V](f(x-y)) = [V](f(y)-f(x)) \geq \min \{ [V](f(x)), [V](f(y)) \} = \min \{ [M](x), [M](y) \}$, which implies that $[M](x-y) \geq \min \{ [M](x), [M](y) \}$, for all x and y in R . Now, $[M](xy) = [V](f(xy)) = [V](f(y)f(x)) \geq \min \{ [V](f(x)), [V](f(y)) \} = \min \{ [M](x), [M](y) \}$, which implies that $[M](xy) \geq \min \{ [M](x), [M](y) \}$, for all x and y in R . Hence $[M]$ is an interval valued fuzzy subring of the ring R .

In the following Theorem \circ is the composition operation of functions:

2.5 Theorem: Let $[M]$ be an interval valued fuzzy subring of a ring H and f is an isomorphism from a ring R onto H .

Then $[M] \circ f$ is an interval valued fuzzy subring of R .

Proof: Let x and y in R and $[M]$ be an interval valued fuzzy subring of the ring H . Then we have, $([M] \circ f)(x-y) = [M](f(x-y)) = [M](f(x) - f(y)) \geq \min \{ [M](f(x)), [M](f(y)) \} \geq \min \{ ([M] \circ f)(x), ([M] \circ f)(y) \}$, which implies that $([M] \circ f)(x-y) \geq \min \{ ([M] \circ f)(x), ([M] \circ f)(y) \}$. Then we have, $([M] \circ f)(xy) = [M](f(xy)) = [M](f(x)f(y)) \geq \min \{ [M](f(x)), [M](f(y)) \} \geq \min \{ ([M] \circ f)(x), ([M] \circ f)(y) \}$, which implies that $([M] \circ f)(xy) \geq \min \{ ([M] \circ f)(x), ([M] \circ f)(y) \}$. Therefore $([M] \circ f)$ is an interval valued fuzzy subring of a ring R .

2.6 Theorem: Let $[M]$ be an interval valued fuzzy subring of a ring H and f is an anti-isomorphism from a ring R onto H . Then $[M] \circ f$ is an interval valued fuzzy subring of R .

Proof: Let x and y in R and $[M]$ be an interval valued fuzzy subring of the ring H . Then we have, $([M] \circ f)(x-y) = [M](f(x-y)) = [M](f(y)-f(x)) \geq \min \{ [M](f(x)), [M](f(y)) \} \geq \min \{ ([M] \circ f)(x), ([M] \circ f)(y) \}$, which implies that $([M] \circ f)(x-y) \geq \min \{ ([M] \circ f)(x), ([M] \circ f)(y) \}$. Then we have, $([M] \circ f)(xy) = [M](f(xy)) = [M](f(y)f(x)) \geq \min \{ [M](f(x)), [M](f(y)) \} \geq \min \{ ([M] \circ f)(x), ([M] \circ f)(y) \}$, which implies that $([M] \circ f)(xy) \geq \min \{ ([M] \circ f)(x), ([M] \circ f)(y) \}$. Therefore $([M] \circ f)$ is an interval valued fuzzy subring of R .

2.7 Theorem: Let $[M]$ be an interval valued fuzzy subring of a ring R , $[M]^+$ be an interval valued fuzzy set in R defined by $[M]^+(x) = [M](x) + [1] - [M](0)$, for all x in R , where 0 is the identity element. Then $[M]^+$ is an interval valued fuzzy subring of the ring R .

Proof: Let x and y in R . We have, $[M]^+(x-y) = [M](x-y) + [1] - [M](0) \geq$

$\min \{ [M](x), [M](y) \} + [1] - [M](0) = \min \{ \{ [M](x) + [1] - [M](0) \}, \{ [M](y) + [1] - [M](0) \} \} = \min \{ [M]^+(x), [M]^+(y) \}$. Therefore, $[M]^+(x-y) \geq \min \{ [M]^+(x), [M]^+(y) \}$, for all x and y in R . Similarly, $[M]^+(xy) = [M](xy) + [1] - [M](0) \geq \min \{ [M](x), [M](y) \} + [1] - [M](0) = \min \{ \{ [M](x) + [1] - [M](0) \}, \{ [M](y) + [1] - [M](0) \} \} = \min \{ [M]^+(x), [M]^+(y) \}$. Therefore, $[M]^+(xy) \geq \min \{ [M]^+(x), [M]^+(y) \}$, for all x and y in R . Hence $[M]^+$ is an interval valued fuzzy subring of the ring R .

2.8 Theorem: Let $[M]$ be an interval valued fuzzy subring of a ring R , $[M]^+$ be an interval valued fuzzy set in R defined by $[M]^+(x) = [M](x) + [1] - [M](0)$, for all x in R , where 0 is the identity element. Then there exists 0 in R such that $[M](0) = [1]$ if and only if $[M]^+(x) = [M](x)$.

Proof: It is trivial.

2.9 Theorem: Let $[M]$ be an interval valued fuzzy subring of a ring R , $[M]^+$ be an interval valued fuzzy set in R defined by $[M]^+(x) = [M](x) + [1] - [M](0)$, for all x in R , where 0 is the identity element. Then there exists x in R such that $[M]^+(x) = [1]$ if and only if $x = 0$.

Proof: It is trivial.

2.10 Theorem: Let $[M]$ be an interval valued fuzzy subring of a ring R , $[M]^+$ be an interval valued fuzzy set in R defined by $[M]^+(x) = [M](x) + [1] - [M](0)$, for all x in R , where 0 is the identity element. Then $([M]^+)^+ = [M]^+$.

Proof: Let x and y in R . We have, $([M]^+)^+(x) = [M]^+(x) + [1] - [M]^+(0) = \{ [M](x) + [1] - [M](0) \} + [1] - \{ [M](0) + [1] - [M](0) \} = [M](x) + [1] - [M](0) = [M]^+(x)$. Hence $([M]^+)^+ = [M]^+$.

2.11 Theorem: Let $[M]$ be an interval valued fuzzy subring of a ring R . Then

$[M]^0$ is an interval valued fuzzy subring of the ring R .

Proof: For any x in R , we have $[M]^0(x-y) = [M](x-y) / [M](0) \geq [1 / [M](0)] \min \{ [M](x), [M](y) \} = \min \{ [[M](x) / [M](0)], [[M](y) / [M](0)] \} = \min \{ [M]^0(x), [M]^0(y) \}$. That is $[M]^0(x-y) \geq \min \{ [M]^0(x), [M]^0(y) \}$, for all x and y in R . Similarly, $[M]^0(xy) = [M](xy) / [M](0) \geq [1 / [M](0)] \min \{ [M](x), [M](y) \} = \min \{ [[M](x) / [M](0)], [[M](y) / [M](0)] \} = \min \{ [M]^0(x), [M]^0(y) \}$. That is $[M]^0(xy) \geq \min \{ [M]^0(x), [M]^0(y) \}$, for all x and y in R . Hence $[M]^0$ is an interval valued fuzzy subring of the ring R .

2.12 Theorem: Let $(R, +, \cdot)$ be a ring. $[M]$ is an interval valued fuzzy subring of R if and only if $[M]^c$ is an interval valued anti-fuzzy subring of R .

Proof: Suppose $[M]$ is an interval valued fuzzy subring of R . For all x and y in R , we have $[M](x-y) \geq \min \{ [M](x), [M](y) \}$, which implies that $[1] - [M]^c(x-y) \geq \min \{ \{ [1] - [M]^c(x) \}, \{ [1] - [M]^c(y) \} \}$, which implies that $[M]^c(x-y) \leq [1] - \min \{ [1] - [M]^c(x), [1] - [M]^c(y) \}$, which implies that $[M]^c(x-y) \leq \max \{ [M]^c(x), [M]^c(y) \}$. Also, $[M](xy) \geq \min \{ [M](x), [M](y) \}$, which implies that $[1] - [M]^c(xy) \geq \min \{ \{ [1] - [M]^c(x) \}, \{ [1] - [M]^c(y) \} \}$, which implies that $[M]^c(xy) \leq [1] - \min \{ [1] - [M]^c(x), [1] - [M]^c(y) \}$, which implies that $[M]^c(xy) \leq \max \{ [M]^c(x), [M]^c(y) \}$. Thus $[M]^c$ is an interval valued anti-fuzzy subring of R . Converse also can be proved similarly.

2.13 Theorem: Let $[M]$ be an interval valued fuzzy subring of a ring R , then the pseudo interval valued fuzzy coset $(a[M])^p$ is an interval valued fuzzy subring of the ring R , for every a in R .

Proof: Let $[M]$ be an interval valued fuzzy subring of the ring R . For every x and y in R , we have, $((a[M])^p)(x-y) =$

$p(a)[M](x-y) \geq p(a) \min \{ [M](x), [M](y) \} = \min \{ p(a)[M](x), p(a)[M](y) \} = \min \{ ((a[M])^p)(x), ((a[M])^p)(y) \}$.
 Therefore, $((a[M])^p)(x-y) \geq \min \{ ((a[M])^p)(x), ((a[M])^p)(y) \}$, for x and y in R . And, $((a[M])^p)(xy) = p(a)[M](xy) \geq p(a) \min \{ [M](x), [M](y) \} = \min \{ p(a)[M](x), p(a)[M](y) \} = \min \{ ((a[M])^p)(x), ((a[M])^p)(y) \}$. Therefore, $((a[M])^p)(xy) \geq \min \{ ((a[M])^p)(x), ((a[M])^p)(y) \}$, for x and y in R . Hence $(a[M])^p$ is an interval valued fuzzy subring of the ring R .

2.14 Theorem: Let $(R, +, \cdot)$ be a ring. If $[M]$ is an interval valued fuzzy subring of R , then $x + [M] = y + [M]$ if and only if $[M](x - y) = [M](0)$, where 0 is the identity element. In that case $[M](x) = [M](y)$.

Proof: Given $[M]$ is an interval valued fuzzy subring of R . Suppose that $x + [M] = y + [M]$, which implies that $(x + [M])(x) = (y + [M])(x)$, which implies that $[M](x - x) = [M](x - y)$, which implies that $[M](0) = [M](x - y)$. Conversely, assume

that $[M](x - y) = [M](0)$, then $(x + [M])(z) = [M](z - x) = [M](z - x + y - y) \geq \min \{ [M](z - y), [M](0) \} = [M](z - y) = (y + [M])(z)$, which implies that $(x + [M])(z) \geq (y + [M])(z)$ -----(1). Now, $(y + [M])(z) = [M](z - y) = [M](z - y + x - x) \geq \min \{ [M](z - x), [M](0) \} = [M](z - x) = (x + [M])(z)$, which implies that $(y + [M])(z) \geq (x + [M])(z)$ ----- (2). From (1) and (2) we get, $x + [M] = y + [M]$.

2.15 Theorem: Let $(R, +, \cdot)$ be a ring. Let $[M]$ is an interval valued fuzzy subring and $x, y, u,$ and v be any elements in R , if $x + [M] = u + [M]$ and $y + [M] = v + [M]$, then $(x + y) + [M] = (u + v) + [M]$.

Proof: Given $[M]$ is an interval valued fuzzy subring of R . By Theorem 2.14, $[M](x - u) = [M](y - v) = [M](0)$. We get, $[M](x + y - u - v) = [M](x - u + y - v) \geq \min \{ [M](x - u), [M](y - v) \} = [M](0)$. Again, by Theorem 2.14, $(x + y) + [M] = (u + v) + [M]$.

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