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Research Paper

Exact Eigenstates of Relativistic Spin-Half Particles in A Radially Varying Magnetic Field Having Azimuthal Symmetry

Physics

KEYWORDS : Eigenstates; Radially decreasing magnetic field; Azimuthal symmetry; Dirac equation.

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ABSTRACT

We discuss exact eigenvalues and eigenfunctions for a relativistic spin-half charged particle in a radially decreasing magnetic field having azimuthal symmetry. The possible applications are mentioned.

Introduction

The exact energy eigenvalues and eigenfunctions of a charged particle in a uniform magnetic field are well known [1]. The energy eigenvalues are known as Landau levels [2]. Significant relativistic corrections to these non-relativistic solutions are derived [3]. Since realistic magnetic fields are inhomogeneous, it is important to find exact eigenvalues and eigenfunctions for inhomogeneous magnetic fields. The exact eigenvalues and eigenfunctions for a non-relativistic charged particle in radially decreasing magnetic field having azimuthal symmetry are available [4]. However it is interesting and important to work out eigenvalues and eigenfunctions of a charged particle having relativistic speeds. Energy eigenvalues and eigenfunctions of a relativistic spin-less charged particle in a radially varying magnetic field having azimuthal symmetry are derived [5]. However it is equally important to work out Eigenstates of spin-half charged particles having relativistic speeds. In this article we discuss energy eigenvalues and eigenfunctions of a relativistic spin-half charged particle in a radially varying magnetic field having azimuthal symmetry.

The Eigenvalue Equation

The Dirac equation in its standard form could be written as

$$i\hbar\frac{\partial\psi}{\partial t} = \left[c\vec{\alpha}.\vec{p} + \beta mc^2\right]\psi \qquad \text{.....................(1)}$$

In the presence of electromagnetic potentials, we can write Dirac equation as

$$\left\{\vec{\alpha}.\left[c\vec{p} - e\vec{A}\right] + \beta mc^2\right\}\psi = \left[E - e\Phi\right]\psi \qquad \text{......(2)}$$

For a pure magnetic field acting along z direction and whose strength decreases radially outwards, we can choose [4],

$\Phi = 0,\ A_Z = 0, \alpha_x = \sigma_x, \alpha_y = \sigma_y,\ \beta = \sigma_Z.$

Where $\vec{\alpha}$ and β are well-known Pauli matrices

$$\alpha_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \alpha_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Since the potential has azimuthal symmetry, the Dirac equation in two space dimensions could be written as

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.(cp_x - eA_x)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.(cp_y - eA_y)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}mc^2\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \text{.......................................(3)}$$

On simplification we get

$$(cp_x - eA_x)\psi_2 - i(cp_y - eA_y)\psi_2 = (E - mc^2)\psi_1 \qquad \text{.....................(4)}$$

And

$$(cp_x - eA_x)\psi_1 + i(cp_y - eA_y)\psi_1 = (E + mc^2)\psi_2 \qquad \text{....................(5)}$$

Substituting for ψ_2 in equation (4) and replacing momentum by its operator we obtain equation in ψ_1 as

$$\left[-i\hbar c\frac{\partial}{\partial x} - eA_x - \hbar c\frac{\partial}{\partial y} + ieA_y\right]\times\left[-i\hbar c\frac{\partial}{\partial x} - eA_x + \hbar c\frac{\partial}{\partial y} - ieA_y\right]\psi_1 = (E^2 - m^2c^4)\psi_1 \qquad \text{........................(6)}$$

We consider the particle in a magnetic field acting along z direction, whose strength decreases radially outwards. The vector potential $\vec{A}$ will have only azimuthal component, since the magnetic field B_Z has only azimuthal symmetry.

We assume

$$B_Z = \frac{B_0}{r^{2\alpha}}$$

Where $r^2 = x^2 + y^2$.

Using this we find x and y components of the potential as

$$A_x = -\frac{B_0}{2(1-\alpha)}\left(\frac{y}{r^{2\alpha}}\right)$$

and $$A_y = \frac{B_0}{2(1-\alpha)}\left(\frac{x}{r^{2\alpha}}\right)$$

For $\alpha \neq 1$, the eigenvalues and eigenfunctions of a relativistic spin-half particle in a vector potential $\vec{A}$ are obtained by solving Dirac equation. Substituting the x and y components of the potential in equation (6) and on simplification the equation reduces to

$$-\hbar^2c^2\nabla^2\psi_1 + \frac{ie\hbar cB_0}{(1-\alpha)}\frac{1}{r^{2\alpha}}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\psi_1 + \frac{e\hbar cB_0\alpha}{(1-\alpha)}\frac{\psi_1}{r^{2\alpha}} + \frac{e^2B_0^{\ 2}}{4(1-\alpha)^2 r^{2\alpha-1}}\psi_1 = (E^2 - m^2c^4)\psi_1 \qquad \text{........................(7)}$$

Changing over to cylindrical coordinates using

$x = r\cos\phi,\ y = r\sin\phi$ and $r^2 = x^2 + y^2$

We get

$$L_z = -i\hbar\frac{\partial}{\partial\phi} \text{ and } \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

Thus the equation (7) becomes

$$\left(-\hbar^2c^2\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{r}\frac{\partial}{\partial r}\right\} + \frac{ie\hbar cB_0}{(1-\alpha)}\frac{1}{r^{2\alpha}}\left\{\frac{\partial}{\partial\phi}\right\} + \frac{e\hbar cB_0\alpha}{(1-\alpha)r^{2\alpha}}\right)\psi_1 + \frac{e^2B_0^2}{4(1-\alpha)^2r^{2\alpha-1}}\psi_1 = \left(E^2 - m^2c^4\right)\psi_1 \quad \text{.......................................(8)}$$

The above equation could be separated into radial and angular components by the method of separation of variables by choosing

$$\psi(\theta,\phi) = R(r)\Phi(\phi)$$

We get, the equation for Φ as

$$\frac{d^2\Phi}{d\phi^2} + l^2\Phi = 0$$

The equation for Φ can be easily solved by considering the periodicity in Φ. This implies the solution as

$$\Phi = \frac{1}{\sqrt{2\pi}}e^{il\phi}$$

Where $l = 0,1,2,3$....................

The radial equation will be

$$\frac{d^2R}{dr^2} - \frac{l^2}{r^2}R + \frac{1}{r}\frac{dR}{dr} + \frac{2eB_0l}{\hbar c}\frac{R}{r} - \frac{e^2B_0^2}{\hbar^2c^2}R - \frac{eB_0}{\hbar c}\frac{R}{r} + \left(\frac{E^2 - m^2c^4}{\hbar^2c^2}\right)R = 0 \quad \text{................(9)}$$

Where we have taken $\alpha = \frac{1}{2}$.

Solution for the radial equation

The radial equation takes the form

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left[\frac{\Omega}{4a^2} - \frac{l^2}{\rho^2} + \frac{2l-1}{\rho} - 1\right]R = 0 \quad \text{...(10)}$$

putting

$$a = \frac{eB_0}{2\hbar c},\ \Omega = \frac{E^2 - m^2c^4}{\hbar^2c^2}$$

and introducing the variable $\rho = 2ar$.

Now with the substitution

$$\xi = 2\sqrt{1 - \frac{\Omega}{4a^2}}\,\rho.$$

The radial equation reduces to

$$\frac{d^2R}{d\xi^2} + \frac{1}{\xi}\frac{dR}{d\xi} + \left[\frac{\dfrac{2l-1}{2\sqrt{1-\dfrac{\Omega}{4a^2}}}}{\xi} - \frac{1}{4} - \frac{l^2}{\xi^2}\right]R = 0 \quad \text{..(11)}$$

Comparing the above equation with equation 16.31 of Schiff Quantum mechanics [6], wherein the radial equation is reduced to the following form by using the separation of variables in parabolic form

$$\frac{d^2f}{d\xi^2} + \frac{1}{\xi}\frac{df}{d\xi} + \left[\frac{\lambda_1}{\xi} - \frac{1}{4} - \frac{m^2}{4\xi^2}\right]f = 0 \quad \text{......................................(12)}$$

We identify

$$\lambda_1 = \frac{2l-1}{2\sqrt{1-\dfrac{\Omega}{4a^2}}} \text{ and } l^2 = \frac{m^2}{4}.$$

The Schiff's solution to equation (12) is

$$f(\xi) = e^{-\frac{1}{2}\xi}\ \xi^{\frac{1}{2}|m|}\ L(\xi)$$

On physical grounds, the solution to $L(\xi)$ that is suitable is such that

$$\lambda_1 - \frac{1}{2}(|m|+1) = n_1$$

Where n_1 is an integer.

This gives the energy eigenvalues, when

$$\frac{1}{2}|m| = |l|.$$

Therefore

$$\frac{l - \frac{1}{2}}{\sqrt{1 - \dfrac{\Omega}{4a^2}}} - \frac{1}{2}(2|l|+1) = n$$

Where n is an integer.

Substituting for Ω and taking $n = n_1 + l$,

we obtain energy eigenvalues of relativistic spin-half charged particle in a radially decreasing magnetic field as

$$E_{n,l} = \left[m^2c^4 + e^2B_0^2 - \frac{e^2B_0^2\left(l - \frac{1}{2}\right)^2}{\left(n + \frac{1}{2}\right)}\right]^{\frac{1}{2}} \quad \text{..(13)}$$

Using the binomial expansion the above expression could be expanded as

$$E_{n,l} = mc^2 + \left[\frac{e^2B_0^2}{2mc^2} - \frac{e^2B_0^2\left(l - \frac{1}{2}\right)^2}{2mc^2\left(n + \frac{1}{2}\right)^2}\right] - \frac{e^4B_0^4}{8m^3c^6\left(n + \frac{1}{2}\right)^4}\left[\left(n + \frac{1}{2}\right)^2 - \left(l - \frac{1}{2}\right)^2\right]^2 + \ldots\ldots \quad \text{...(14)}$$

In the above equation the first term represents rest energy, second term represents non relativistic energy term [4] and the third term represents first order relativistic correction and so on.

The radial wave function could be written as

$$R_{(nl)}(\xi) = Ne^{-\frac{1}{2}\xi}\,\xi^{|l|}\,L_{n+2|l|}^{2|l|}(\xi)$$

Where $L_{n+2|l|}^{2|l|}(\xi)$ are associated Laguerre polynomials [7] and N is normalisation constant.

Results and discussions

We have obtained the exact energy eigenvalues and eigenfunctions of a relativistic spin-half charged particle in a radially varying magnetic field having azimuthal symmetry. Since the relativistic correction obtained contributes significantly when the strength of the magnetic field is considerably high. Our results may find significant number of applications while describing the dynamics of relativistic spin-half charged particle in the strong magnetic fields generated by astronomical objects like neutron stars, Black holes etc. Also these eigenstates are important in describing the dynamics of the relativistic particles in suitable current loops carrying high currents.

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