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Automatic Boundedeness of Homomorphisms on Bornological Algebras

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ABSTRACT

In this paper, we study the problem of the automatic boundedness of homomorphisms in certain complete bornological algebras. We also extend from the results, known in the case normalized with the bornological algebras.

Introduction

In [6], Sinclair studied the necessary condition for continuity of homomrphisms on a Banach space. The aim of the present paper is to extend some of this results in case of bornological algebra and consequently obtains some techniques to answer the boundness problem for linear operators. We extend naturally the notion of separting space of some linear operator S between (bvs) X and (bsv) Y. The notion of separting space characterizes the continuity of linear operator. The utility of separating space comes owing to the fact that a linear operator is bounded if, and only if, its separating space is reduced to the singleton {0} ([6]). The characterization of the semi-simplicity of an associative unital algebra A brings back its left ideals under investigation. Indeed, it is well-known which A is semi simple if, and only if, the intersection as of its maximum left ideals is tiny room to {0}. In this work, we define too for an algebra in involution * (A, *) a concept that we call * semi simplicity, it rests on the study of certain ideals. The interest thus is to restrict with a family of the ideals instead of considering all the left ideals. This concept of *semi simplicity will contribute also under investigation of the problem of the automatic continuity of the linear operators in the topological algebras.

Preliminaries

Throughout this work, the algebras considered are supposed to be complex, associative, unital and not necessarily commutative; of type M1. Recall that a bornology on a set X is a family β of subset X such that β is a convering of X, hereditary under inclusion and stable under finite union. The pair (X, β) is called bornological set. A subfamily β' of β is said to be base of bornology β, if every element of β is contained in an element of β'. Let X and Y be two bornological set, a map X of Y is called bounded if the image of every bounded subset of X is bounded in Y. A bornology β on IK-vector space E is said to be vector bornology on E if the maps (x,y) ⟶ x+y and (λ,x) ⟶ λ.x are bounded. We called a bornological vector space (b.v.s) any pair (X, β) consisting of a vector space E end a vector bornology β on E.

A vector Bornology on a vector space is called a convex vector bornology if it is stable under formation of convex hull. A bornological vector space is said a convex bornological vector space (c.b.s) if it bornology is convex. A (b.v.s) space is called of type M_1 if, it satisfies the countability of Mackey: *For every sequence of bounded $(B_k)_k$ in , there exists a sequence of scalars $(\lambda k)_k$ such that $\bigcup_{k=0}^{\infty} \lambda_k B_k$ is bounded in E*

Observe that every Banach space is multiplicative convex bornological complete algebra of type M_1 A sequence $(x_n)_n$ in bornological vector space (bvs) E is said Mackey-convergent (or converge bornological to 0) if there exists a bounded set B ⊂ E such that

$$\forall \varepsilon \succ 0 \quad \exists n_0 \in IN \; n \geq n_0 \text{ implies } x_n \in \varepsilon B$$

If is (cbs), then $(x_n)_n$ is Mackey-convergent to 0 if there exists a bounded disk B ⊂ E such that $(x_n)_n$ ⊂ E and $(x_n)_n$ converge to 0 in E_B, where (E_B, p_B) is the vector space spanned by B and endowed with semi-norm p_B gauge of B. A (bvs) is separated if, and only if, for every bounded disk B the space (E_B, p_B) is a normed space. A set B in (bvs) E is said M-closed (or b-closed) if every sequence $(x_n)_n$ ⊂ B Mackey convergent in E its limit belongs in B .

Let E a separated (bvs) and let F be a subspace of E, the bornological quotient space E/F is separated, if and only if, F is b-closed in E. Let E be a (cbs) and A a disk in E. A is called a completant disk if the space (E_B, p_B) spanned by A and semi-normed by the gauge of A is a Banach space. A (cbs) E is called a complete convex bornological vector space if its bornology has a base consisting of completant disk.

A *-ring is a ring with a map * : A → A that is an involution. More precisely, * is required to satisfy the following properties: $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, $1^* = 1$, $(x^*)^* = x$ for all x,y in A . This is also called an involutive ring, and ring with involution. Elements such that $x^* = x$ are called self-adjoint. Also, one can define *-versions of algebraic objects, such as ideal and subring, with the requirement to be *-invariant: $x \in I \Rightarrow x^* \in I$ and so on. A *-alge~~bra~~ A is a *-ring, with involution *, such that $(\lambda x)^* = \lambda x^*$ $\forall \lambda \in IK, x \in A$. An algebra A is called simple if it has no proper ideals. An *-algebra A is called *-simple if it has no proper *ideals An associative unital algebra A over a field IK is **semi simple** if its Jacobson radical is trivial. An associative unital *-algebra A over a field IK is ***semi simple** if its Jacobson *radical (Rad_*) is trivial.

Definition 1.1 [7] : Let X and Y two bornological vector spaces (b.v.s) and T a linear map of X in Y. we called separating space of T, the subset of Y denotes by σ (T):

$$\sigma(T) = \{y \in Y / \exists (x_n)_n \subset X : x_n \to^M 0 \text{ et } T(x_n) \to^M y\}$$

Definition 1.2 [7] : Let X and Y two bornological vector spaces (b.v.s) and T a linear map of X in Y. We called separating space of T in X, the subset of X denotes by σ' (T):

$$\sigma'(T) = \{x \in X / \exists (x_n)_n \subset X : x_n \to^M 0 \text{ et } T(x_n) \to^M T(x)\}$$

Proposition 1.1 [7] Let X and Y two convex bornological spaces (c.b.s). Then, the separating space of any linear map T: X ⟶ Y is a b-closed subspace of Y.

Proposition 1.2 [7] Let X and Y two c.b.s and T: X ⟶ Y a linear map. Then, we have:

I) σ (T) = {0} if, and only if, the graph of T b-is closed.

II) If R and S are two linear operators of X and Y respectively and if, TR = ST, then:

S (σ (T)) C σ (T).

Corollary 1.1 [7] Let X be a complete (c.b.s) and Y a c.b.s such that its bornology has a net. Let T: X ⟶ Y be a linear map. Then T is bounded if, and only if, σ (T) = {0}.

Proposition 1.3 [7] Let X and Y two b.c.s of the M1 type and Z a separated b.v.s . Suppose that X is complete and the bornology of Y has a net. Let S: X ⟶ Y be linear and R: Y ⟶ Z be bounded linear map. Then:

I) RS is bounded if, and only if, Rσ (S) = {0}.

II) $[R\sigma(S)]^{(1)}$ = σ(RS).

Proposition 1.4 Let X and Y two convex bornological spaces. Then, separating space σ' (S) of any linear map S: X ⟶ Y is a b-closed subspace of X.

Proof:
Evidently σ› (S) is a subspace vector of X.

Let σ (S) the separating space of S in Y and Q: Y ⟶ Y/σ (S) the canonical surjection.

Q (σ (S)) = {0}, then, according to the previous Proposition, QS is bounded.

We have: σ'(S) = S^{-1}[σ(S)] = Ker(QS) = $(QS)^{-1}$({0})

As σ (S) b-is closed, Y/ σ (S) is separated. Then, σ'(S) = S^{-1}[σ(S)] is a b-closed of X.

Proposition 1.5 [7] Let X and Y two c.b.s of type M1 and S: X ⟶ Y a linear map. Suppose that X is complete and the bornology of Y has a net. Let X_0 and Y_0 two subspace b-closed of X and Y respectively such that $S(X_0)$ C Y_0 Let S_0 : X/ X_0 ⟶ Y/ Y_0 defined by : $S_0(x + X_0) = S(x) + Y_0$ is bounded if, and only if, σ(S) C Y_0

AUTOMATIC BOUNDEDNESS
We propose to establish certain results concerning the automatic boundedness of surjective homomorphisms (Or has with dense range) in the multiplicative Convex bornological complete algebra (m.c.b.a). a m.c.b.a with involution * denoted m.c.b.*a

Proposition 2.1 Let X and Y two b.c.s and S: X ⟶ Y a linear map. Suppose that X is complete and the bornology of Y is has a net.

Let σ (S) the separating space of S in Y (resp. σ' (S) the separating space of S in X). That is to say S_0 : X/ σ'(S) ⟶ Y/ σ(S) defined by: S_0(x + σ'(S)) = S(x) + σ(S) for any x in X. Then S_0 is bounded.

Proof
As σ'(S) = S-1[σ(S)], S(σ'(S)) C σ(S)). Moreover, σ (S) (resp. σ' (S)) is a b-closed vector subspace of Y (resp. X). Consequently, S_0 is bounded (Proposition (1.5)).

Proposition 2.2 Let T a homomorphism of a m.c.b.a A in a m.c.b.a B then. if T is surjective (or with dense range), then the separating space σ (T) of T is an ideal of B.

Proof
Let β (resp. β') is pseudo-base formed by discs, idempotents, ordered pre and filtering increasing of A (resp. B). Evidently σ (T) is a vectorial subspace of B. Suppose that T is surjective. Let b ∈ σ(T) and b' ∈ B. As b∈ σ(T), there exists a sequence $(a_n)_n$ C A such that: $a_n \to^M 0$ et $T(a_n) \to^M b$. $a_n \to^M 0$, then there exists a B_1 an element of β such that $(a_n)_n$ in A_{B1} and $P_{B1}(a_n) \to 0$. Moreover, b' ∈ B, then there exists a' ∈ A such that T(a') = b'. We have: a' ∈ A, then there exists B_2 an element of β such that a ∈ A_{B2}. However β is filtering increasing, from where there exists an element B_3 of β such that : A_{B1} C A_{B3} and A_{B2} C A_{B3} and $P_{B1} \le P_{B3}$. Then we have : a ∈ A_{B3} which is an Banach algebra, and $P_{B3}(a_n) \to 0$. It thus results from it that $P_{B3}(aa_n) \le P_{B3}(a) P\beta_3(a_n) \to 0$. Consequently, $aa_n \to^M 0$. By the same reasoning, it is shown that $b'T(a_n) \to^M b'b$. In addition, $T(a'a_n) = T(a')T(a_n) = b' T(a_n) \to^M b' b$. what implies that b'b ∈ σ(T). In the same way, it is shown that bb' ∈ σ(T). Suppose now that T is with dense range. Let b ∈ σ(T) and b' ∈ B. b ∈ σ(T), there exists a sequence $(a_n)_n$ C A such that: $a_n \to^M 0$ et $T(a_n) \to^M b$. According to what precedes, one a: for any N, $T(a_n)b$ ∈ σ(T) and $bT(a_n)$ ∈ σ(T). Moreover, $bT(a_n) \to^M bb'$ et $T(a_n)b \to^M b'b$. However σ (T) is M-closed (Proposition 3.1), it thus results from it that b'b ∈ σ(T) et bb' ∈ σ(T). Consequently, σ (T) is an ideal of B.

Proposition 2.4 Let A a complete (m.c.b.a). and $(x_n)_n$ be a sequence of elements of A which converges bornological to an element x of A. Let V a neighborhood of 0 in ℂ. Then it exists a positive entirety n such that: $Sp(x_n)$ C Sp(x) + V, pour tout n ≥ N.

Proof
Let β is pseudo-base formed by completant, idempotents, ordered pre and filtering increasing discs of A. Let us suppose that there exists an open neighborhood of 0 in ℂ such that, for any n it exists n ≥ N such as $Sp(x_n) \not\subset$ Sp(x) + V. Let $(z_n)_n$ subsequence of de $(x_n)_n$ and $(\lambda_n)_n$ C ℂ such that: $\lambda_n \in Sp(z_n)$ and $\lambda_n \notin Sp(x) + V$, for any n, Like $(z_n)_n$ converges bornological to x, there exists an element B_1 de β such that: $z_n - x \in A_{B1}$ and $P_{B1}(z_n - x) \to 0$. Let B_2 an element of β such that x ∈ A_{B2}. As β is filtering increasing, there exists a B_3 element of β such that: A_{B1} C A_{B3} and A_{B2} C A_{B3} and $P_{B1} \le P_{B3}$. According to this precedes, $(z_n)_n$ C A_{B3} and $P_{B3}(z_n - x) \to 0$. Thus the sequence $(z_n)_n$ is bounded in the Banach space A_{B3}. As $Sp(z_n)$ C $Sp_{B3}(z_n)$, it results from it that $\lambda_n \in Sp_{B3}(z_n)$ for any n. Consequently $|\lambda| \le P_{B3}(z_n)$ for any n. From where, the sequence $(\lambda_n)_n$ is bounded. Thus one can extract subsequence $(\mu_n)_n$ from $(\lambda_n)_n$ which converges to an element μ. On other hand, μ ∉ Sp(x) + V and consequently μ ∉ Sp(x). it is followed from there that, x – μe is invertible. Moreover, $(\mu e - x) - (\mu_n e - z_n) = (\mu - \mu_n)e + (x - z_n)$ converges bornological to 0, which implies $\mu_n e - z_n$ is invertible for n rather large. From where, $\mu_n \notin Sp(z_n)$.

Proposition 2.4 Let T a homomorphism of a complete (m.c.b.a) A in a complete (m.c.b.a) B. If b ∈ σ(T), then Sp (b) is a subset of ℂ which contains 0.

Proof
Let us suppose that 0 ∉ Sp(b). As If b ∈ σ(T), there exists a sequence $(a_n)_n$ C A such that $a_n \to^M 0$ in A and $T(a_n) \to^M b$ in B. Let us choose compact neighborhood V of 0 in ℂ such that 0 ∉ Sp(b) + V. Thus, for n rather large, $Sp(a_n) \cap$ Sp(b) V = ϕ. However $Sp(T(a_n))$ C $Sp(a_n)$. Then, for n rather large, $Sp(T(a_n) \cap Sp(b) + V = \phi$, which contradicts the Proposition (2. 3).

Definition 2.1 Let T a linear map to a complete c.b.s E in a complete c.b.s F. We called that T is dense range in F if $(T(E))^{(1)}$ = F.

Proposition 2.5 Let T a homomorphism of a complete (m.c.b.a) A in a complete (m.c.b.a) B at countable base. If B is simple and am T is surjective (Or with dense range), then T is bounded.

Proof
Let σ (T) the separating ideal of T in B which is simple, therefore: σ (T) = {0} or σ (T) = B. If σ (T) = B, then $1_B \in$ σ(T). According to the Proposition (2.2), 0 ∈ Sp(1_B), What impossible. From where σ (T) = {0}. Consequently, T is bounded.

Proposition 2.6 Let A an algebra *simple which is not simple. Then, there exists a subalgebra simple unit I of A such as A = I $\oplus$ I^*.

Proof
Let I a proper ideal of A. . I $\cap$ I^* is one *ideal, therefore I $\cap$ I^* = {0} or I $\cap$ I^* = A. If I $\cap$ I^* = A. If I $\cap$ I^* = A then I = A, which is absurd. From where I $\cap$ I^* = {0}. There is also I + I* is one *ideal, then I + I* = {0} or I + I* = A. If I + I* = {0}, then I = {0}, which contradicts the fact that I am *proper*. Therefore, A = I $\oplus$ I^*. Let J a ideal of A such as J $\subseteq$ I. According to what precedes, A = J $\oplus$ J^*. Let i ∈ I, then there exists j, j' ∈ J such that i = j + j'*. However i - j = j'* ∈ I $\cap$ I^* = {0}, from where i = j, therefore I = J. Consequently, I am a minimal ideal of A. Let J an ideal of I, then J is an ideal of A. Indeed, let a ∈ A and j ∈ J , then it exists i,i'* such that a = i + i'*. From where aj = (i + i'*)j = ij + i'*j. However, i'*j ∈ I^*I $\subseteq$ I $\cap$ I^* = {0}, consequently aj = ij ∈ J. As I am a minimal ideal, then J = {0} or I = J. Thus, I am simple subalgebra. On other hand, I am unital and if 1 indicates the unit of A, then there exists e, e' ∈ I such that 1 = e + e'*. Let x ∈ I, we are: x = x1 = xe + x e'*, but x - xe = x e'* ∈ I $\cap$ I^* = {0}, from where x = xe. In the same way, we checked that x = xe. Consequently, I am unital of unit e.

Theorem 2.1 Let T a homomorphism of a complete (m.c.b.a) A in a complete (m.c.b.a) B at countable base. If B is *simple and if T is surjective (or with dense range), then T is bounded.

Proof
Let β pseudo-bases formed by completant and idempotents discs. As B is an algebra *simple, there exists subalgebra simple unital I of B such as: B = I $\oplus$ I^* (Proposition 2.6), then of following algebraic isomorphism: : I ≈ B/I^*, we deduces that I am a maximum ideal of B. From where I (resp. I*) M-closed in B ([1] Proposition II-1.3). Consequently, according to the Proposition (1, [2]), I (resp. I*) is a complete c.b.s. Let β_I the set defined by: β_I = {B $\cap$ I/ B ∈ β}. βI is a base of m.c.b.a on I. Consequently, I (resp. I*) is complete m.c.b subalgebra of type M_1 of B. Consider: $Pr_1 : B \rightarrow I$ (resp. $Pr_2 : B \rightarrow I^*$) the canonical projection of B on I (resp. of B on I*). Since : Pr_1 (resp. : Pr_2) is bounded epimorphism (or with dense range if T is), then according to the Proposition (2.5) $Pr_1 \circ T$ (resp. $Pr_2 \circ T$) is bounded. Consequently, T = (Pr_1 + Pr_2) $\circ$ T = $Pr_1 \circ T + Pr_2 \circ T$ is bounded.

Proposition 2.7 Let M an ideal *maximal of a m.c.b.*a A. then M is a M-closed ideal of A.

Proof
If M is a maximum ideal of A, then M is M- closed (Proposition II-1.3 [1]). If not, there exists a maximum ideal N of A such as M = N $\cap$ N^* * (Proposition 2.8 [8]). However N (resp. N*) M-is closed, we deduces that M is M-is closed in A.

Theorem 2.2 Let T a homomorphism of a complete (m.c.b.a) A in a complete (m.c.b.a) B at countable base. If B is *semi-simple and if T is surjective (or with dense range), then T is bounded.

Proof
Let β pseudo-bases formed by completant and idempotent discs. Let M a *maximal ideal of B. Then involution * induced an involution on B/M, also noted * defined by: * (a+M) = *(a) +M. Let β_M the set defined by β_M = {B+M/B ∈ β}. β_M is a pseudo-bases formed completant and idempotents discs on B/M, therefore B/M is a m.c.b.*a *simple. As M is M-closed (Proposition 2.7), it results from it that B/M is a complete c.b.s ([2] Proposition 2). Let us consider the canonical surjection Q : B $\rightarrow$ B/M. As Q is bounded, then according to the theorem (2.1), homomorphism Q $\circ$ T is bounded. Consequently, σ (Q $\circ$ T) = {0}. Such as σ (Q $\circ$ T) = [Q (σ (T)]$^{(1)}$ (Proposition3.4 [7]), it is deduced that, Q (σ (T) = {0}. From where σ (T) $\subseteq$ M . Since M is arbitrary, then (σ (T) $\subseteq$ $\cap$ M = Rad_* (B) ={0}. Consequently, T is bounded.

Corollary 2.1 Let an *-algebra *semi-simple A. If β1 and β2 two complete bornology of m.c.b.a at countable bases on A, then β1 = β2.

Proof
It is enough to apply the previous theorem to the identity of A.

Theorem 2.3 Let (A, β) an m.c.b.*a at bases countable *simple. Then the involution * is automatically bounded.

Proof
Let β* a set defined by: β* = {B*/B β}. β* is a complete bornology of m.c.b.a on A. indeed: Let x an element of A, we can write x as: x = y + iz, where y and z two element self-adjoint of A. Let B_1 and B_2 two elements of β such that y ∈ B_1 and z ∈ B_2. We have, y ∈ B_1^* and z ∈ B_2^*. Therefore, x ∈ B_1^*+i B_2^* = (B_1 – iB_2)*, but or B_1 – iB_2 ∈ β, there exists an element B of β such as B_1 – iB_2 = B, which implies, x ∈ B*. Consequently, β* recovers A. Let B_1^* an element of β* and B_2 part of A such that: B_2 ⊂ B_1^*. We have , B_2^* ⊂ B_1, as B_1 ∈ β, it is followed from there that B_2^* ∈ β, from where B_2 ∈ β*. Let B_1^* and B_2^* two elements of β*. Let us show that B_1^* ∪ B_2^* ∈ β*. We have, B1* ∪ B2* $\subseteq$ (B1 ∪ B_2)*. as B_1 ∪ B_2 ∈ β, then (B_1^* ∪ B_2^*)* ∈ β, consequently, B_1^* ∪ B_2^* β* . Let us show that β* is a vectorial bornology. Let B_1^* and B_2^* two elements of β* and λ ∈ ℂ. We have: B_1^* + B_2^* =(B_1 + B_2)* and λ B_1^* = (λ B_1)*. As B_1 + B_2 and λ B_1 are in β, then B_1^* + B_2^* and λ B_1 are in β*. β* is a bornology of c.b.a. Let B_1^* and B_2^* two elements of β*. Then, $B_1^*B_2^*$ =(B_2 B_1)*. However, B_2 B_1 ∈ β, from where B_1^* B_2^* ∈ β*. Remain to show that β* is a complete bornology of m.c.b.a at countable base. Let B a completant and idempotents discs of β, then B* is also a idempotant disc. If x ∈ AB, then we have:

P_B(x)= inf { λ > 0/ x ∈ λ B} =inf { λ > 0/ x^* ∈ $\overline{\lambda}$ B^* = λ B^*} = P_{B^*}(x^*)

Since (A_B, P_B) is a Banach algebra, it thus results from it that (A_B^*, P_B^*) is too. It is easy to check that β* is at countable base on A. According to the previous theorem, β =β*, consequently, the involution * is bounded.

Corollary 2.3 Let T a homomorphism of a Banach algebra A in a Banach *algebra B. If B is *semi-simple and if T is surjective (or with dense range), then T is bounded.

Proof
As any Banach algebra is a m.c.b.a. complete at countable base and of type M1, the corollary rises immediately from the theorem (2.3)

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