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Non Bivalent Pigeonhole Principle

Mathematics

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ABSTRACT *Usefulness and significance are the key concerns while framing any logic. Now, as the complexity of a system increases our ability to make certain and yet significant statements about problems which are fuzzy in character diminishes. But in the real world there exist much fuzziness. The knowledge, which is uncertain, imprecise, vague, inexact or probabilistic in nature, demands equal attention. Gone are the days of barter system, now you can earn a lot with the click of your mouse on a stock exchange without the exchange of a single penny. Fuzzy logic, which is non bivalent in character, enables the human reasoning capabilities to be applied to artificial knowledge-based systems with high level computation. This paper is devoted in finding how fuzziness can genereralise pigeonhole principle.*

1. INTRODUCTION

"In my view, in years to come fuzzy logic – its name notwithstanding – is certain to grow in visibility, importance and acceptance. There is a basic reason. In essence, fuzzy logic is the logic of classes with unsharp boundaries. In the world of human cognition, classes with unsharp boundaries are the rule rather than exception. What this implies is that it is only a matter of time before the essentiality of fuzzy logic becomes apparent not only in science but in most domains of human thought and culture." ……. Lotfi A. Zadeh

Man has come of age and now he needs a different vision; he needs a different kind of approach, only then the tomorrow can be possible. Most of our traditional tools of modeling, reasoning and computing are crisp, discrete and bivalent in character and now they needs to be viewed under an electron microscope. Usefulness and significance are the key concerns while framing any logic. Now, as the complexity of a system increases our ability to make certain and yet significant statements about problems, especially which are fuzzy in character, diminishes. The real world is fuzzy. So the knowledge which is uncertain, imprecise, vague, inexact or probabilistic in nature cannot be discarded as such and rather it demands equal attention.

The sets of best players in a game, intelligent students in a class and beautiful girls in a competition, are often regarded as not well defined sets because they do not fulfill the criterion of belongingness. In classical notion of set theory the membership of elements in relation to a set is assessed in crisp terms according to a bivalent condition. An element either belongs or does not belong to the set, the boundary of the set is crisp. As a further development of classical set theory, fuzzy set theory permits the gradual assessment of the membership of elements in relation to a set; this is described with the aid of a membership function. Such a set is characterized by a membership (characteristic function) which assigns to each object a grade of membership ranging between 0 and 1.

2. Fuzzy Pigeonhole Principle

If n sets contain n + 1 or more distinct elements in all, at least one of the sets contains two or more elements or if one place n + 1 pigeons into n pigeonholes, then at least one hole will occupy two or more pigeons. More formally, there does not exist an injective function on finite sets whose codomain is smaller than its domain. If one place infinitely many pigeons into finite number of pigeonholes then at least one hole is occupied by infinitely many pigeon. Now what if the set is not finite? Does the pigeonhole principal is still valid? If yes, under what conditions? For answering these questions a peep into analysis involving compact sets and fuzzy set theory is required.

2.1 Compact Metric Space

The metric space (X, d) is compact if for every every open cover $\{G_\lambda : \lambda \in \Lambda\}$ of X there exist finite subcover of open sets i.e. if $X = \{G\lambda : \lambda \in \Lambda\}$, there exist finitely many sets $G\lambda 1$, $G\lambda 2$,…., $G_{\lambda n}$ among G_λ's such that X = Union of $G_{\lambda 1}$, $G_{\lambda 2}$,……., $G_{\lambda n}$.

2.2 Cluster Point

$x \in X$ is said to be a cluster point or subsequent limit of $\{x_n\}$ if there exist s subsequence of $\{x_n\}$ converging to x or more precisely for any $\in > 0$ (however small) and any positive integer m, there exist an integer $n \geq m$ such that $d(x, x_n) < \in$. In other words x will be a cluster point of the sequence $\{x_n\}$ iff every open sphere centred at x contains infinitely many terms of the sequence.

2.3 Fuzzy Set

TLet X be a space of points (objects), with a generic element of X denoted by x. A *fuzzy set* A in X is characterized by a membership (characteristic) function fA(x) which associates with each point in A a real number in the interval [0, 1].

A fuzzy set A can be written as

$A = \{(x, f_A(x))\}, x \in X, f_A: X \rightarrow [0, 1]$.

Let X be a finite set and $\{x_n\}$ be a sequence in X. Then there exist $x \in X$ such that x is visited infinitely many times or in other words if X is a finite set of pigeonholes and we have infinite pigeons, then at least one pigeonhole will occupy infinitely many pigeons. So pigeonhole principle is valid in this case. But if the set X is infinite say [0, 1], then infinite pigeons can be placed at 1, 1/2, 1/3, 1/4, 1/5, ………., 1/n, ……..which are infinite positions. So there does not exist $x \in X$ in this example which is visited many times. Hence pigeonhole principle fails in this case. But an interesting thing emerges here, the sequence {1/n} converges to 0 i.e. infinite pigeons are near a single pigeonhole at 0. So if we introduce the concept of fuzzinesss or closeness in precise pigeonholes we can again recover the principle.

2.4 Fuzzy Pigeonhole Principle

For any positive $\in > 0$, however small (degree of fuzziness), and for any sequence of points $\{x_n\}$ in compact metric space (X, d) we can find a point x in X, such that infinitely many points in the sequence satisfy $d(x_n, x) < \in$ i.e. infinite pigeons $\{x_n\}$ instead of visiting the pigeonhole x precisely are as near to x as we wish or infinitely many pigeons visit the open sphere centred at x.

If $\{x_n\}$ is a finite sequence in X, then one point is visited infinitely many times and the principal holds. Now suppose for $\{x_n\}$ having infinite range the principle does not hold i.e. such x does not exist. Then for any $x \in X$ we can find an $\in$ or an open neighbourhood N_x of x which contains no point of $\{x_n\}$ other than (possibly) x. Now the collection $\{N_x : x \in X\}$ covers X implying there exist finitelv many points ^{x1}, ^{x2}, ……., ^{xm} in X such that $X = N^{x1} \cup N^{x2} \cup ……. \cup N^{xm}$ (X is compact). Since $\{x_n\}$ $\in\in$ X implying $\{x_n\}$

has at most n points which is a contradiction.

3. APPLICATIONS

Many applications of fuzzy set theory have accumulated since its inception. There is hardly any field where it has not made its inroads. It is employed in a wide variety of products, like cameras, appliances, hybrid engines and transmissions, copiers, elevators, blood pressure meters, etc. Even in social sciences like psychology, for instance, fuzzy set-based theories of perception (e.g., Oden & Massaro, 1978, and Sequelae) and memory (Massaro, Weldon, & Kitzis, 1991) have appeared. Likewise, in sociology and political science, fuzzy sets have been advocated by Ragin (2000) as "diversity oriented" research which strengthens the connection between theory and data analysis. Bárdossy and Duckstein (1995) have used it for regional planning. Now suppose we have a chess board with two of the diagonally opposite corners removed and we want to cover the board with pieces of domino whose size is exactly two board squares. Two diagonally opposite squares on a chess board are of the same color. Therefore, when these are removed, the number of squares of one color exceeds by 2 the number of squares of another color. Every placement of domino pieces establishes a 1-1correspondence between the set of white squares and the set of black squares. If the two sets have different number of elements, then, by the Pigeonhole Principle, no 1-1 correspondence between the two sets is possible. Take another case from analysis. Let X = [0, 1] , d the usual metric i.e. d(x, y) = ∈x - y∈ and $\{x_n\} = \{1/n\}$ be a sequence in X. Then for any positive ∈ (however small) there exist x ∈ X (0 in this case) such that $d(x_n, x) < \in$ or 1/n <∈. In other words, we divide [0, 1] into smaller intervals [0, ∈) , [∈, 2∈) , [2∈, 3∈) , ...

Total number of such intervals =

$$\begin{cases} \frac{1}{\epsilon} & if \frac{1}{\epsilon} \in N \text{ (The set of Natural Numbers)} \\ \left[\frac{1}{\epsilon}\right] & if \frac{1}{\epsilon} \text{ does not belong to } N \end{cases}$$

finite in both case.

For example for ∈ = 1/3, 1/4, 1/5 the subintervals are

[0, ∈) , [∈, 2∈) , [2∈, 3∈) ;

[0, ∈) , [∈, 2∈) , [2∈, 3∈) , [3∈, 4∈) ;

[0, ∈) , [∈ 2∈) , [2∈, 3∈) , [3∈, 4∈) , [4∈, 5∈)

and for ∈ = 3.4 subintervals are

[0, ∈) , [∈, 2∈) , [2∈, 3∈) , [3∈, 1]

The length of these subintervals is less than ∈. Thus [0, 1] can be divided into finite intervals but the sequence $\{x_n\}$ is infinite implying there exist a subinterval where infinite points of the sequence can land. For the sequence

$$\begin{cases} \frac{1}{n} & if\ n\ is\ odd \\ 1 - \frac{1}{n} & if\ n\ is\ even \end{cases}$$

0 and 1 are the required points. Take another example, let X = [-1, 1] and $\{x_n\} = \{(-1)^n\}$, -1 and 1 are the required points. Since S(1, ∈) , open sphere centered at 1, contains infinite 1 for any ∈.

Consider the sequence $\{x_n\} = \{1/n\}$ in X = N. Here 0 does not belong to N. So fuzzy pigeonhole principle fails. Moreover the set N is not compact. For N to be a compact set, every open cover of N must have finite subcover of open sets. Suppose N is compact. Now the collection { (0, n) : n ∈ N } covers N since N ⊆ ⋃ { (0, n) : n ∈ N }. There must exist a finite subcover satisfying

$N \subseteq (0, n_1) \cup (0, n_2) \cup (0, n_3) \cup \ldots\ldots\ldots \cup (0, n_k)$

Let $n_0 = \max \{ n_1, n_2, n_3, \ldots\ldots\ldots, n_k \}$, Then n_0 does not belong to the finite subcollection implying N is not compact.

4. CONCLUSION

This logic looks simple but its applications are not as simple. This paper is devoted in finding how fuzziness can genereralise pigeonhole principle and how it can be applied to solve problems from different streams. The open question still remains how to characterize the space in which infinite pigeonhole principle holds.

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