

On Square Sum Chain Graphs whose Blocks are Complete Graphs



Mathematics

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ABSTRACT

A (p,q) -graph G is said to be square sum, if there exists a bijection $f:V(G) \rightarrow \{0,1,2,\dots,p-1\}$ such that the induced function $f^*:E(G) \rightarrow \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. Barrientos [2] defines a chain graph as one with blocks $B_1, B_2, B_3, \dots, B_m$ such that for every i , B_i and B_{i+1} have a common vertex in such a way that the block cut point graph is a path. In this paper the problem of which chain graph whose blocks are complete graphs are square sum is studied.

1. Introduction

Unless mentioned otherwise, by a graph we shall mean in this paper a finite, undirected, connected graph without loops or multiple edges. Labeling of a graph G is an assignment of integers either to the vertices or edges or both subject to certain conditions. A dynamic survey to know about numerous graph labeling method is regularly updated by J.A Gallian [7]. Terms not defined here are used in the sense of Harary [10]. Acharya and Germina defined a square sum labeling of a (p,q) -graph G [1,8] as follows.

Definition 1.1 A (p,q) -graph G is said to be square sum, if there exists a bijection $f:V(G) \rightarrow \{0,1,2,\dots,p-1\}$ such that the induced function $f^*:E(G) \rightarrow \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective.

We will frequently use the following result which are proved in [5,6].

Theorem 1.2 [5]: The complete graph K_n is square sum if and only if $n \leq 5$.

Algorithm in [6] is used to find the edges having same label in K_n , $n \geq 6$. In this paper we are inclined to bring out a few newer aspects of square sum Husimi chain. Barrientos [2] defines a chain graph as one with blocks $B_1, B_2, B_3, \dots, B_m$ such that for every i , B_i and

B_{i+1} have a common vertex in such a way that the block cut point graph [10] is a path. We will denote the chain graph with n blocks and the sequence of n blocks of complete graphs $K(a_1), K(a_2), K(a_3), \dots, K(a_n)$ by $CK(n; (a_1, a_2, a_3, \dots, a_n))$. We will assume that $a_i \geq 2$, $i=1,2,\dots,n$. If $a_1 = a_2 = \dots = a_n = 2$, then $CK(n; (2,2,\dots,2)) = P_{n+1}$. It is well known that P_n is square sum. Several classes of graph had shown to be square sum [1,5]. We refer to $CK(n; (a_1, a_2, \dots, a_n))$, $a_i \geq 2$ as a Husimi Chain (Chain graph) [8]. In this paper we propose to investigate the existence of square sum labeling for certain classes of chain graphs whose blocks are complete graphs.

Theorem 1.3. The triangular snake $S_{3,n}$ obtained from the path $P_n = (u_0, u_1, \dots, u_n)$ by replacing the edge $u_i u_{i+1}$ by the cycle $(u_i u_{i+1} v_{i+1} u_i)$, $0 \leq i \leq n-1$ is square sum.

Proof: Here $|V(S_{3,n})| = 2n+1$.

Define $f: V(S_{3,n}) \rightarrow \{0,1,\dots,2n\}$ by $f(u_i) = 2i$, $0 \leq i \leq n$, $f(v_i) = 2i-1$, $0 \leq i \leq n$. Clearly f is injective. With the above defined vertex label, the induced edge labels can be arranged in strictly increasing order and hence are distinct.

If $a_1 = a_2 = \dots = a_n = 3$, then $CK(n; 3, 3, \dots, 3)$ is the triangular snake and hence by Theorem 1.3, it is square sum.

2 One Point union of two complete graphs

If $n=2$, then $CK(2; (a_1, a_2)) \cong K(a_1) \bullet K(a_2)$, is the one point union of two complete graphs. We assume $a_1 \leq a_2$.

Theorem 2.1 For $n=2$: $a_1=2$; $CK(2; (a_1, a_2))$ is square sum if and only if $a_2 \leq 8$.

Proof. For $n \leq 7$, label the vertices of K_2 by 0,1 and n -vertices of K_n are labeled by 1,2... n such that the cut point is labeled by 1. For $n=8$, label the vertices of K_2 by 4,0 and n -vertices of K_8 are labeled by 0,1,2,3,5,6,7 and 8 such that the cut point is labeled by 0. The square sum labeling of $CK(2; (a_1, a_2))$, $a_1=2$ and for $a_2 = 3,4,5,6,7,8$ is depicted in Figure 1.

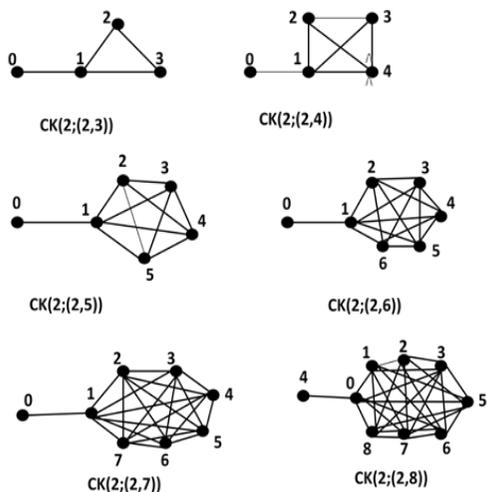


Figure 1: Square sum labeling of $CK(2; (a_1, a_2))$, when $a_1=2$ and $a_2 \leq 8$.

Now, let $a_2 \geq 9$ and $f: V(CK(2; (2, a_2))) \rightarrow \{0,1,2,\dots, a_2\}$ be a square sum labeling such that $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(CK(2; (2, a_2)))$ are all distinct. Since a_2 is complete graph we always have pair of edges e_1 and e_2 or e_3 and e_4 or e_5 and e_6 in $CK(2; (2, a_2))$ such that $f^*(e_1) = 0^2 + 5^2 = 25$, $f^*(e_2) = 3^2 + 4^2 = 25$, $f^*(e_3) = 1^2 + 8^2 = 65$, $f^*(e_4) = 4^2 + 7^2 = 65$, $f^*(e_5) = 2^2 + 9^2 = 85$, $f^*(e_6) = 6^2 + 7^2 = 85$, is a contradiction to the fact that f is square sum.

Theorem 2.2 . For $n=2$: $a_1=3$; $CK(2; (a_1, a_2))$ is square sum if and only if $a_2 \leq 9$.

Proof: The square sum labeling of $CK(2; (3, a_2))$, for $a_2 \leq 9$ and is depicted in Figure 2.

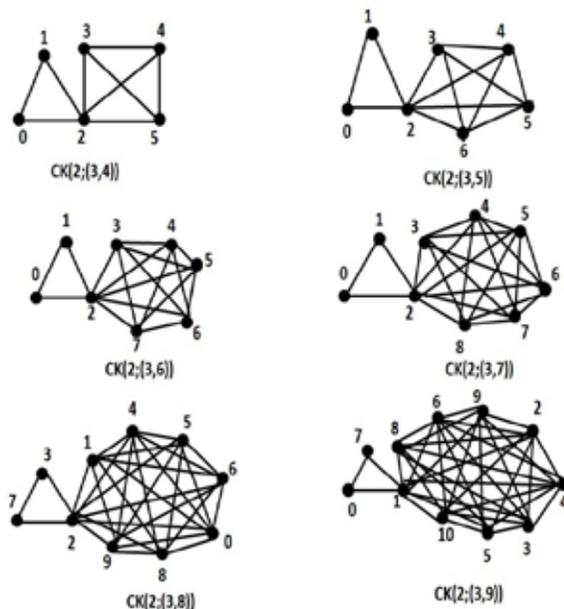


Figure 2: Square sum labeling of $CK(2; (3, a_2))$ for $a_2 \leq 9$

Let $CK(2; (3, a_2))$, $a_2 \geq 10$ and $f: V(CK(2; (3, a_2))) \rightarrow \{0, 1, 2, \dots, a_2 + 1\}$ be a square sum labeling such that $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(CK(2; (3, a_2)))$ are all distinct. For $n \geq 10$, by using algorithm in [5] we will get edges having same label, a contradiction to the fact that f is square sum.

Theorem 2.3 For $n=2$: $a_1=4$, the graph $CK(2; (a_1, a_2))$ is square sum if and only if $a_2 \leq 8$.

Proof: The square sum labeling of $CK(2; (4, a_2))$, for $a_2 \leq 8$ is depicted in Figure 3.

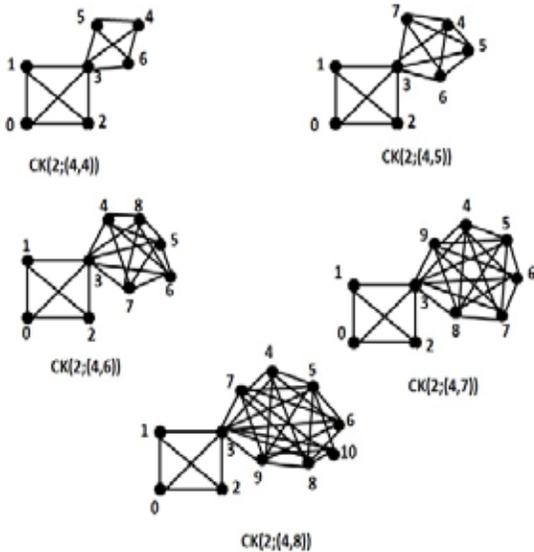


Figure 3: Square sum labeling of $CK(2; (4, a_2))$ for $a_2 \leq 8$

Let $CK(2; (4, a_2))$, $a_2 \geq 9$ and $f: V(CK(2; (4, a_2))) \rightarrow \{0, 1, 2, \dots, a_2 + 2\}$ be a square sum labeling such that $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(CK(2; (4, a_2)))$ are all distinct. For $a_2 \geq 9$, by using algorithm in [5] we will get edges having same label. This is a contradiction to the fact that f is square sum.

3 Chain graph with 3 blocks

Theorem 3.1 For $n=3$: $a_1=2, a_2=3$, the chain graph $CK(3; (a_1, a_2, a_3))$ is square sum for $a_3 \leq 8$.

Proof: The square sum labeling of $CK(3; (a_1, a_2, a_3))$, for $n=3, a_1=2, a_2=3$ is depicted in Figure 4.

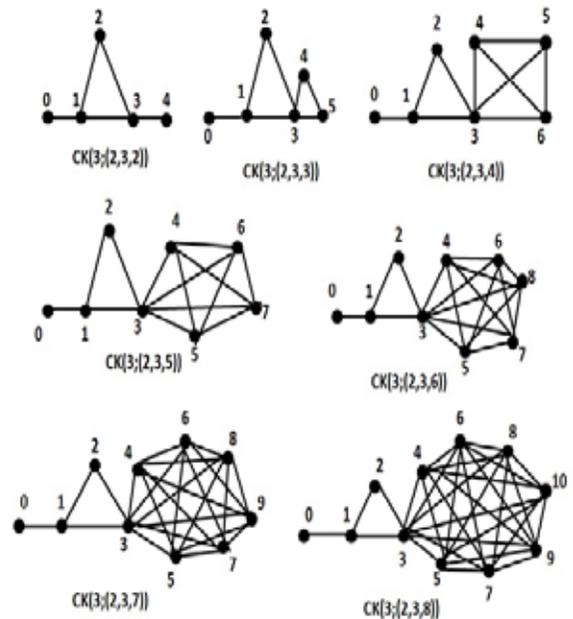


Figure 4: Square sum labeling of $CK(3; (a_1, a_2, a_3))$ for $a_1=2, a_2=3, a_3 \leq 8$

Remark 3.2 For $a_3 \geq 9$, we will always get

two edges of same label using Algorithm [5].

Theorem 3.3 For $n=3$: $a_1=2, a_2=4$, the chain graph $CK(3; (a_1, a_2, a_3))$ is square sum for $a_3 \leq 9$.

Proof: The square sum labeling of $CK(3; (a_1, a_2, a_3))$, for $a_1=2, a_2= 4$ and $a_3 \leq 9$ is depicted in Figure 5.

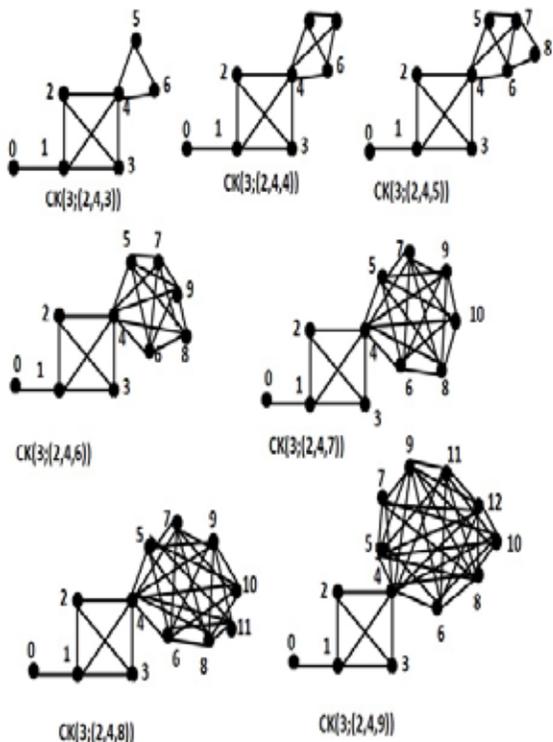


Figure 5: Square sum labeling of $CK(3; (a_1, a_2, a_3))$ for $a_1=2, a_2=4, a_3 \leq 9$.

Remark 3.4 For $a_3 \geq 10$ we will always get two edges of same label using Algorithm [5].

Theorem 3.5 For $n=3: a_1=2, a_2=5$, the chain graph $CK(3; (a_1, a_2, a_3))$ is square sum for $a_3 \leq 9$.

Proof: The square sum labeling of $CK(3; (a_1, a_2, a_3))$, for $a_1=2, a_2= 5$ and $a_3 \leq 9$ is depicted in Figure 6.

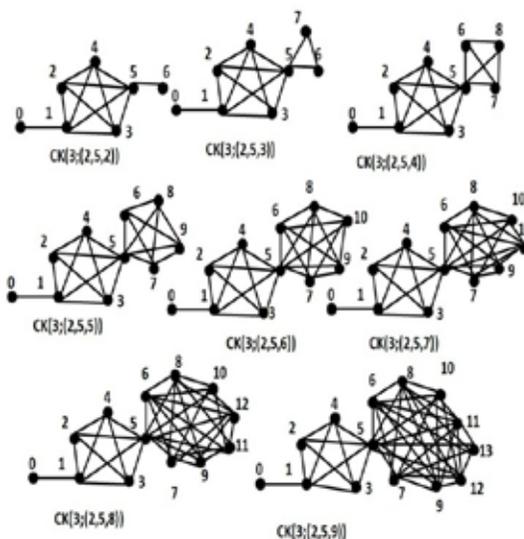


Figure 6: Square sum labeling of $CK(3; (a_1, a_2, a_3))$ for $a_1=2, a_2=5, a_3 \leq 9$.

Remark 3.6 For $a_3 \geq 10$ we will always get two edges of same label using Algorithm [5].

Observation 3.7 For $n=3: a_1=2, a_2=6$, the chain graph $CK(3; (a_1, a_2, a_3))$ is square sum for $a_3 \leq 11$.

Observation 3.8 For $n=3: a_1=2, a_2=7$, the chain graph $CK(3; (a_1, a_2, a_3))$ is square sum for $a_3 \leq 12$.

Observation 3.9 For $n=3: a_1=2, a_2=8$, the chain graph $CK(3; (a_1, a_2, a_3))$ is square sum for $a_3 \leq 11$.

Theorem 3.10 For $n=3: a_1=3, a_2=9$, the chain graph $CK(3; (a_1, a_2, a_3))$ is square sum for $a_3 \leq 4$.

Proof: The square sum labeling of $CK(3; (a_1, a_2, a_3))$ for $a_1=3, a_2= 9$ and $a_3 \leq 4$ is depicted in Figure 7.

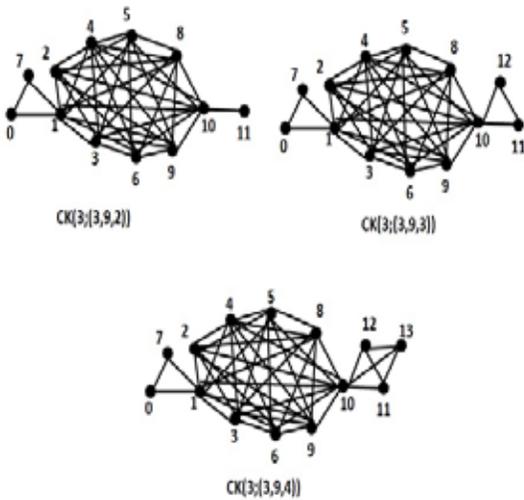


Figure 7: Square sum labeling of $CK(3; (a_1, a_2, a_3))$ for $a_1=3, a_2=9, a_3 \leq 4$.

Remark 3.11 For simplicity of presentation, we will represent a square sum chain graph whose blocks are complete graphs K_t , in the form of t-tuples and call each t-tuple a sequence and square sum labeling is written as a string of such sequences corresponding to the complete blocks in the Husimi Chain. For example, consider the square sum Husimi chain $CK(4;(2,3,4,4))$ we represent the square sum labeling of it as the string $(0,1)(1,2,3)(3,4,5,6)(6,7,8,9)$.

Theorem 3.12 For $n \geq 3$ the chain graph $CK(n; (2,8,4,4\dots))$ are square sum.

Proof: We start with $CK(3;(2,8,4))$ together with one of the square sum labeling is given by the string $(4,0)(0,1,2,3,5,6,7,8)(8,9,10,11)$. The fourth

sequence of the square sum chain $CK(4; (2,8,4,4))$ is obtained by adding 3 to each number of the last sequence $CK(3;(2,8,4))$. Thus $CK(4; (2,8,4,4))$ is given by the string $(4,0) (0,1,2,3,5,6,7,8) (8,9,10,11) (11,12,13,14)$, which is square sum. The fifth sequence of the square sum chain $CK(5; (2,8,4,4,4))$ is obtained by adding 3 to each number of the last sequence $CK(3;(2,8,4,4))$. Thus $CK(5; (2,8,4,4,4))$ is given by the string $(4,0)(0,1,2,3,5,6,7,8)(8,9,10,11)(11,12,13,14)(14,15,16,17)$, which is square sum. Continuing in this manner, we get a sequence of square sum chain graph $CK(n; 2,8,4,4,\dots)$. So we can take the block K_4 as many times as we wish so that the graph $CK(n; 2,8,4,4,\dots)$ is square sum.

Theorem 3.13 . For $n \geq 3$ the chain graph $CK(n; (3,9,4,4\dots))$ are square sum.

Proof: We start with $CK(3;(3,9,4))$ together with one of the square sum labeling is given by the string $(0,7,1)(1,2,3,4,5,6,8,9,10)(10,11,12,13)$. The fourth sequence of the square sum chain $CK(4; (3,9,4,4))$ is obtained by adding 3 to each number of the last sequence $CK(3;(3,9,4))$. Thus $CK(4; (3,9,4,4))$ is given by the string $(0,7,1) (1,2,3,4,5,6,8,9,10) (10,11,12,13) (13,14,15,16)$,

which is square sum. The fifth sequence of the square sum chain $CK(5; (3,9,4,4,4))$ is obtained by adding 3 to each number of the last sequence $CK(4;(3,9,4,4))$. Thus $CK(5; (3,9,4,4,4))$ is given by the string $(0,7,1) (1,2,3,4,5,6,8,9,10) (10,11,12,13) (13,14,15,16) (16,17,18,19)$, which is square sum. Continuing in this manner, we get a sequence of square sum chain graph $CK(n; 3,9,4,4, \dots)$. So we can take the block K_4 as many times as we wish so that the graph $CK(n; 3,9,4,4, \dots)$ is square sum. (see Fig 8).

Remark 3.14 We start with a square sum chain graph $C_1 = CK(3;(3,9,4))$ and construct the square sum chain graph $C_2 = CK(4;(3,9,4,4))$. From C_2 we construct $C_3 = CK(5;(3,9,4,4,4))$, which is square sum. Proceeding in this manner we get an ascending sequence $C = C_1 \subset C_2 \subset \dots$ where C_i ' are square sum.

Theorem 3.15 . For $n \geq 3$, the chain graph $CK(n;(4,3,4,4, \dots))$, $CK(n;(5,3,4,4, \dots))$, $CK(n; 6,3,4,4, \dots)$ are square sum.

Proof We label the graph by the following ways.

$CK(n; 4,3,4,4, \dots) \text{----} (0,1,2,3)(3,4,5)(5,6,7,8) \text{----}$
 $CK(n; 5,3,4,4, \dots) \text{----} (0,1,2,3,4)((4,5,6)(6,7,8,9) \text{----}$

$CK(n; 6,3,4,4, \dots) \text{----} (0,1,2,3,4,6)(6,5,7)(7,8,9,10) \text{----}$
 $((10,11,12,13) \text{----} \text{-----}$

Remark 3.16 For $n=4$, $CK(4;(4,4,4,4))$ are square sum with the labeling $(0,1,2,3) (3,4,5,6) (6,7,8,9) (9,10,11,12)$. This labeling can be infinitely extended to another square sum labeling for the graph $CK(n;(4,4,4,4, \dots))$.

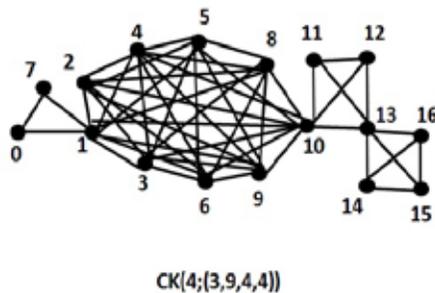
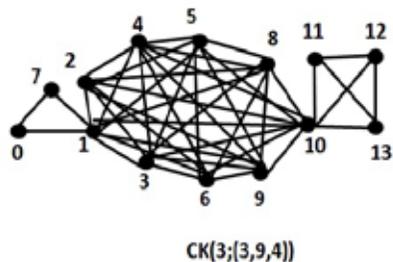


Figure 8: Square sum labeling of $CK(n; (3,9,4,4, \dots, 4))$

We will write $CK(n; a_1, a_2, \dots, a_n)$ as

$CK(n; (a^{[k]}, a_{k+1}, \dots, a_n))$ if $a_1 = a_2 = \dots = a_k = a$.

Theorem 3.17. The chain graph $CK(m+n+1, (4^{[m]}, 3, 4^{[n]}))$ is square sum for all $m, n \geq 0$.

Proof We exhibit square sum labeling for each graph in Figure 9.

Observation 3.18 The chain graph $CK(m+n+1);(4^{[m]},2,3^{[n]})$ is square sum for all $m,n \geq 0$.

Observation 3.19 The chain graph $CK(m, 3^{[m]})$ is square sum for all m .

Observation 3.20 The chain graph $CK(m+n+s);(4^{[m]},2^{[n]},3^{[s]})$ is square sum for all $m,n,s \geq 0$.

We have generated infinitely many classes of square sum Husimi Chain, chain graph whose blocks are complete. It is worth investigating what are the other combinations of blocks other than complete blocks so that the chain graphs are square sum.

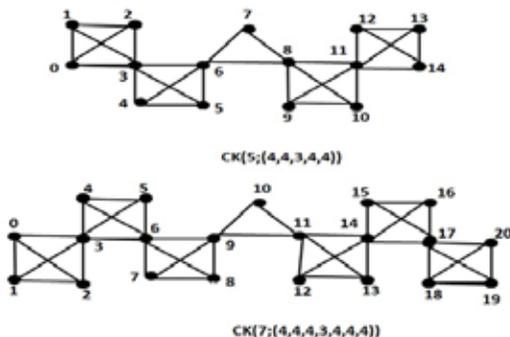


Figure 9

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