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Research Paper

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Bayesian analysis of an n unit system operating in a random environment

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ABSTRACT

The system under consideration is an n unit system operating in a random environment, wherein the environment determines the number of units required for the satisfactory performance of the system at every instant of time. Assuming that the environment is described by a Markov Process, the maximum likelihood and consistent estimators of ρ, the limiting property of the estimator , Uniformly Minimum Variance Unbiased estimator (UMVUE) of ρ^l ($l \geq 1$) and UMVUE of the expected number of operable units in the system are obtained. Further, a 100(1-α) % asymptotic confidence interval for the expected number of operable units in the system, Bayes estimator of ρ under squared error loss, the minimum posterior risk associated with Bayes estimator and the minimum Bayes risk of $\hat{\rho}^B$ are obtained.

1. Summary

In the probabilistic analysis of redundant systems, it is noted that the system reliability depends primarily on three factors. First, it depends upon the failure rates or equivalently the lifetime distributions of the components constituting the system. The reliability of these components is determined by the manufacturer giving due consideration to the cost. Secondly, the system reliability depends upon the system configuration, that is the number of components and the manner in which these components are connected to form the system. Thirdly, the system reliability depends upon the repair maintenance and (or) the corrective or preventive maintenance performed on the system.

A variety of models have been proposed and studied extensively in the past giving due importance to these three factors. However, there is yet another factor, which has not been given its due recognition so far and which affects the system reliability namely the environmental conditions under which the system operates. In certain situations, the environment may determine the number of units required for the satisfactory performance of the system at every instant of time. For example, to increase the thermal power plant availability, an additional induced draft fan (ID fan) may be installed in 200 MW sets, though two ID fans are normally used to handle flue gas and fly ash during full load operation of the plant. i.e., the load on a system may change randomly; see Das and Acharya (1988). Again, in a telecommunication network, the success of sending a message from an origin to a destination depends upon the existence of at least one path connecting the origin with the destination with all units determining the path in the operable state. Hence, the number of units required for sending the message successfully at any time is determined by the availability of units in the intermediate stations and the locations of the origin and destination. Sharafali et al (1988) have considered a two unit system with similar assumption and obtained expressions for the mean time to the first disappointment and expected number of disappointments in an interval. Chandrasekhar and Natarajan (1999) have extended the same to an n unit system with similar assumptions and obtained the expressions for (i) the distribution and the moments of the time to the first disappointment and (ii) the expected number of disappointments over an arbitrary interval (0,t]. Further, Chandrasekhar et al (2005) have studied an n unit system operating in a random environment wherein a unit in standby can fail and obtained the similar results. Also see Limnios and Cocozza (1992).

An attempt is made in this paper to study a system consisting of n units with the assumption that the environment prescribing the number of units required for the satisfactory performance of the system at any time t is governed by a Markov process $\{Y(t): t \geq 0\}$. Reliability theory has attracted researchers all over the world from various disciplines due to many interesting stochastic, operational and statistical problems that naturally arise in the study of several complex systems (e.g. n unit systems, intermittently used systems). A problem that has gained importance due to practical application of systems in Reliability Theory is that of Statistical Inference for failure time and repair time parameters and related parametric functions. However, the problem of maximum Likelihood estimation and Uniform Minimum Variance Unbiased (UMVU) estimation of the ratio of the failure time and repair time parameters $\rho = \frac{\lambda}{\mu}$, estimation of system size distribution and a measure of system performance in the steady state based on a random sample of size N on the number of operable units observed at several sampled time points has not been dealt with for systems operating in random environments.

The outline of this paper is as follows: The model, the assumptions associated with the model, analysis of one unit system and its generalization are discussed in detail in Section 2.

In Section 3, the maximum likelihood estimation and consistent estimator of ρ, the limiting property of $\hat{\rho}$, the UMVUE of ρ^l and UMVUE of the expected number of operable units in the system are obtained. Bayes estimator of ρ under squared error loss, the minimum posterior risk associated with Bayes estimator and the minimum Bayes risk of $\hat{\rho}^B$ are obtained in Section 4. The Model and assumptions are discussed in detail in the following section.

2. The Model and assumptions

The assumptions of the model are given below:

<u>One unit system</u>

i) The system under consideration is a one unit system with constant failure rate λ and constant repair rate μ.

ii) At any instant of time, the system is found in any one of the following mutually exclusive and exhaustive states:
State 0: the system is operating
State 1: the system is failed and is under repair.

iii) The environment process determining the number of units required for the satisfactory performance of the system at any time t is a Markov Process {Y(t): t ≥ 0} with the state space {0,1}. It may be noted that the environment process is independent of the system behaviour.

iv) At time t=0, the unit just begins to operate.

v) When the system fails, it is taken up for repair instantaneously.

2.1 Analysis of the system

Let Q(t) be the number of operable units in the system at time t and $\lim_{t\to\infty} Q(t) = Q$. Clearly, the random variable Q can assume only two values 0 and 1. Further, let $\lim_{t\to\infty} \Pr\{Q(t) = n\} = \Pr\{Q = n\} = p_n$, n = 0, 1. Since the failure time and repair time are exponential with the parameters λ and μ respectively, it is clear that the underlying stochastic process { Q(t) | t ∈ [0,∞) } is a Markov Process with the infinitesimal generator (in fact, an

alternating Renewal Process as the system is either in an up state or down state alternately) given by

$$A = \begin{array}{c} 0 \\ 1 \end{array}\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}. \qquad (2.1)$$

Let $p_i(t)$ be the probability that the system is in state i at time t, i = 0,1 with the initial condition $p_0(0) = 1$. From the infinitesimal generator A given above, we obtain the following system of differential-difference equations:

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) + \mu p_1(t) \quad \text{and}$$

$$\frac{dp_1(t)}{dt} = \lambda p_0(t) - \mu p_1(t) \qquad (2.2)$$

Taking Laplace transforms on both the sides of (2.2) and solving them using the fact that $p_0(0) = 1$ and inverting, it can be shown that

$$p_0(t) = \frac{\mu}{(\lambda+\mu)} + \frac{\lambda}{(\lambda+\mu)} e^{-(\lambda+\mu)t} \quad \text{and}$$

$$p_1(t) = \frac{\lambda}{(\lambda+\mu)} - \frac{\lambda}{(\lambda+\mu)} e^{-(\lambda+\mu)t} \qquad (2.3)$$

The limiting probabilities p_0 and p_1 are determined by allowing $t \to \infty$ on both the sides of (2.3). In otherwords, the limiting probabilities are given by

$$p_0 = \frac{\mu}{(\lambda+\mu)} \quad \text{and} \quad p_1 = \frac{\lambda}{(\lambda+\mu)} \qquad (2.4)$$

2.2 Generalization

Suppose that there are n such identical units undergoing the transactions between upstate and down state independently of each other with the transition structure as described above. Then, the probability distribution of the number of units out of a total of n units in state 0 (operable state) is given by the following Binomial probability law:

$$P(Q=K) = p_k = \binom{n}{k} p_0^k p_1^{n-k}, \quad k = 0,1,2,\ldots,n,$$

$$0 < p_0 < 1, \quad p_1 = (1-p_0) \quad , \qquad (2.5)$$

where p_0 and p_1 are given as in (2.4). By letting $\rho = \frac{\lambda}{\mu}$, we have $p_0 = \frac{1}{(1+\rho)}$, $p_1 = \frac{\rho}{(1+\rho)}$ and (2.5) is simplified to

$$p_k = \binom{n}{k} \left(\frac{1}{1+\rho}\right)^k \left(\frac{\rho}{1+\rho}\right)^{n-k},$$

k=0,1,2,…,n.
(2.6)

In the next Section (2.3), the system description and assumptions of an n unit system operating in a random environment are given.

2.3 n unit system operating in a random environment

i) There are n similar units in the system, which are statistically independent. The probability distribution of k units operable out of n units is given by the Binomial probability law as in (2.5).
ii) The environment process determining the number of units required for the satisfactory performance of the system at any time t is a Markov Process with the state space {0,1,2,..,n}. It may be noted that the environment process is independent of the system behaviour.
iii) If at any time t, Y(t) = i, then i (i=0,1,2,…,n) of n identical units are online (if operable) and the remaining units will be kept as cold standbys. These i units which are online behave like a series system.
iv) Whenever online unit fails, a standby unit if operable is switched online instantaneously.
v) The failed units are taken up for repair in FIFO order. However, a repair for a failed unit cannot commence, when the environment process is in state zero. Repair is perfect.
vi) Whenever the number of units in the operable state is less than the number

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of units required at that instant of time for the satisfactory performance of the system, the system enters the down state.

vii) When the system is in the down state, an operable unit cannot fail.

3. Maximum likelihood estimation of measure of system performance

In this section, we obtain the MLE and consistent estimator of ρ, UMVUE of ρ^l and estimation of the system size distribution. The likelihood function of the number of units $(x_1, x_2, \dots, x_N)$, N ≤ n operable at N different sampled time points $t_1, t_2, \dots, t_N$ is given by

$$L(\rho \mid x_1, x_2, \dots, x_N) = \prod_{i=1}^{N} \left\{ \binom{n}{x_i} \left(\frac{1}{1+\rho}\right)^{x_i} \left(\frac{\rho}{1+\rho}\right)^{n-x_i} \right\}$$

That is

$$L(\rho \mid x_1, x_2, \dots, x_N) = \prod_{i=1}^{N} \left\{ \binom{n}{x_i} \right\} \left(\frac{1}{1+\rho}\right)^{\sum_{i=1}^{N} x_i} \left(\frac{\rho}{1+\rho}\right)^{\sum_{i=1}^{N} (n-x_i)} \qquad (3.1)$$

Clearly, $\left(\frac{\partial \log L}{\partial \rho}\right) = 0$, which implies

$$\hat{\rho} = \frac{(n-\bar{x})}{\bar{x}}, where\ \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i .$$

Hence, the MLE of ρ is given by

$$\hat{\rho} = \frac{(nN-y)}{y}, where\ \ y = \sum_{i=1}^{N} x_i$$

Since $X_1, X_2, \dots, X_N$ are i.i.d Binomial variates each with the parameters $(n, \frac{1}{1+\rho})$, it follows that

$Y = \sum_{i=1}^{N} X_i \sim B(nN, \frac{1}{1+\rho})$ with

$E(Y) = \frac{nN}{(1+\rho)}$ and $Var(Y) = \frac{nN\rho}{(1+\rho)^2}$.

Further,

$$Pr(Y = y) = \binom{nN}{y} \left(\frac{1}{1+\rho}\right)^{y} \left(\frac{\rho}{1+\rho}\right)^{nN-y},$$

$$y = 0,1,2, \dots, nN .$$

$$= f(y; \rho) \quad (say) \qquad (3.2)$$

Since $\hat{\rho}$ is a one-one function of y, $\hat{\rho}$ takes the values $\frac{(nN-y)}{y}, y = 0,1,2, \dots, nN$ with the probability mass function given by

$$Pr[\hat{\rho} = u] = Pr\left[\frac{(nN-Y)}{Y} = u\right]$$

$$= Pr\left[Y = \frac{nN}{1+u}\right]$$

$$= \binom{nN}{\frac{nN}{1+u}} \left(\frac{1}{1+\rho}\right)^{\frac{nN}{1+u}} \left(\frac{\rho}{1+\rho}\right)^{\frac{nNu}{1+u}},$$

$where\ u = \frac{(nN-y)}{y}$.

Now, for large N, $E[\hat{\rho}] \cong \frac{nN - E(Y)}{E(Y)} = \rho$

$$and\ V[\hat{\rho}] \cong \left[\left(\frac{d\hat{\rho}}{dy}\right)^2 Var(Y)\right] \ \ at$$

$$E(Y) = \frac{nN}{(1+\rho)} \ \cong \frac{\rho(1+\rho)^2}{nN} \to 0$$

$as\ \ N \to \infty$.

In otherwords, $\hat{\rho}$ is a consistent estimator of ρ. Further, we obtain the following from (3.1) .

$$\left(\frac{\partial \log L}{\partial \rho}\right) = \frac{-N\bar{x}}{(1+\rho)} + \frac{N(n-\bar{x})}{\rho(1+\rho)} \ ,$$

$$\left(\frac{\partial^2 \log L}{\partial \rho^2}\right) = \frac{N\bar{x}\rho^2 - N(n-\bar{x})(1+2\rho)}{\rho^2(1+\rho)^2}$$

$$and\ E\left(\frac{-\partial^2 \log L}{\partial \rho^2}\right) = \frac{Nn}{\rho(1+\rho)^2} .$$

Hence $Var(\hat{\rho}) = \frac{1}{E\left(\frac{-\partial^2 \log L}{\partial \rho^2}\right)} = \frac{\rho(1+\rho)^2}{nN}$

$= \sigma^2\ (say)$.

Since a consistent solution of the likelihood equation is asymptotically normally distributed about the true value of the parameter, we have

$$\frac{\sqrt{N}(\hat{\rho}-\rho)}{\sigma} = \frac{\sqrt{n}N(\hat{\rho}-\rho)}{(1+\rho)\sqrt{\rho}} \ \ \xrightarrow{d} \ \ N(0,1)$$

$$for\ large\ N\ . \qquad (3.3)$$

3.1 Best unbiased estimation of parametric functions

Initially, we obtain the UMVUE of $\rho^l (l \geq 1)$ in the n unit system operating in a random environment, which is the steady state (stationary) probability of having one or more units operable in the system in order to obtain the UMVU estimation of steady state distribution of system size.

3.1.1 Estimation of ρ^l

Let T_l be the UMVUE of ρ^l , $l \in \{1,2,3,\dots\}$ and is derived by using Lehmann-Scheffe theorem. By reparametrizing the Binomial probability law given in (2.6) with $= \frac{1}{1+\rho}$, we see that the parameter space $\Theta = \{\rho: 0 < \rho < \infty\}$ leads to $\Theta = \{\theta: 0 < \theta < 1\}$ and the corresponding Binomial probability mass function is given by

$$p_k = \binom{n}{k} \theta^k (1-\theta)^{n-k}, \quad k = 0,1,2,\dots,n . \tag{3.4}$$

For this family, the complete sufficient statistic is given by $Y_1 = \sum_{i=1}^{N} X_i$. By Lehmann-Scheffe theorem, the UMVUE of ρ^l is any function $\phi(y_1)$ that is an unbiased estimator of ρ^l. In otherwords, we have $[\phi(Y_1)] = \rho^l = (1-\theta)^l \theta^{-l}$.

That is $\sum_{y_1=0}^{nN} \phi(y_1) \binom{nN}{y_1} \theta^{y_1} (1-\theta)^{nN-y_1}$

$= (1-\theta)^l \theta^{-l}$.

By substituting $\theta_1 = \frac{\theta}{1-\theta}$ and nN=N$_1$ in the above expression, it can be shown that

$$\sum_{y_1=0}^{N_1} \phi(y_1) \binom{N_1}{y_1} \theta_1^{y_1} = \theta_1^{-l} (1+\theta_1)^{N_1} .$$

Equating the coefficient of $\theta_1^{y_1}$ on both the sides and dividing by $\binom{N_1}{y_1}$, we get

$$T_l = \phi(y_1) = \begin{cases} \dfrac{\binom{N_1}{l+y_1}}{\binom{N_1}{y_1}} , & y_1 = 1,2,3,\dots,(N_1 - l), l < N_1 \\ \dfrac{1}{\binom{N_1}{y_1}} , & y_1 = 0, l = N_1 \\ 0 , & l > N_1 \end{cases} \tag{3.5}$$

the UMVUE of ρ^l.

3.1.2 System size distribution

Since a linear combination of UMVUEs is again a UMVUE, the UMVUE of steady state probability distribution of system size in an n unit system operating in a random environment is given by $\hat{p}_k = T_k - T_{k+1}$, where p_k is given by (2.6) and T_l is the UMVUE of ρ^l given by (3.5). In otherwords,

$$\hat{p}_k = \begin{cases} \dfrac{\binom{N_1}{y_1+k}}{\binom{N_1}{y_1}} \dfrac{[2(y_1+k)-(N_1-1)]}{(y_1+k+1)} , & y_1 = 1,2,\dots,(N_1-k-1),(N_1-k);\ N_1 > k+1 \\ 0 & , y_1 = 0,\ N_1 = (k+1) \end{cases}$$

3.1.3 UMVUE of the number of operable units in the system

We, now, derive the UMVUE of measure of system performance given by $E(Q) = \frac{n}{(1+\rho)}$. Since $Y_1 = \sum_{i=1}^{N} X_i$ is a complete sufficient statistic, the UMVUE of $\frac{n}{1+\rho}$ is derived by the application of Lehmann-Scheffe theorem. Since $\rho = \frac{(1-\theta)}{\theta}$, we have $E(Q) = \frac{n}{(1+\rho)} = n\theta$.

Hence, by Lehmann-Scheffe theorem, $\phi(Y_1)$is the UMVUE of nθ, if

$E_\theta\{\phi(Y_1)\} = n\theta,\ \theta \in (0,1)$,

which implies

$$\sum_{y_1=0}^{nN} \phi(y_1) \binom{nN}{y_1} \theta^{y_1} (1-\theta)^{nN-y_1} = n\theta \quad . \tag{3.6}$$

By choosing $\theta_1 = \frac{\theta}{(1-\theta)}$ and nN=N$_1$, (3.6) is simplified to

$$\sum_{y_1=0}^{N_1} \phi(y_1) \binom{N_1}{y_1} \theta_1^{y_1} = n\theta_1 (1+\theta_1)^{N_1-1} \quad . \tag{3.7}$$

Collecting the coefficient of $\theta_1^{y_1}$ on both the sides of (3.7) and dividing by $\binom{N_1}{y_1}$, we get

$$\phi(y_1) = \begin{cases} 0 & , \ y_1 = 0 \\ n\dfrac{\binom{N_1-1}{y_1-1}}{\binom{N_1}{y_1}} & , \ y_1 = 1,2,3,\dots,N_1 \end{cases} \tag{3.8}$$

is the UMVUE of E(Q).

Remark 3.1: Consider

$$\phi(y_1) = \frac{n\binom{N_1-1}{y_1-1}}{\binom{N_1}{y_1}} = \frac{y_1}{N} = \bar{x} \text{ . Clearly, the}$$

UMVUE of nθ is the sample mean estimator $\bar{x}$, as it should be, for nθ is the mean of the pmf given in (3.4).

3.1.4 Confidence limits for the expected number of operable units in the system.

Let $X_1, X_2, X_3, \dots, X_N$ and $Y_1, Y_2, Y_3, \dots, Y_N$ be two random samples each of size N chosen from two different exponential lifetime and exponential repair time populations with the parameters λ and μ respectively. It is clean that $E(\bar{X}) = 1/\lambda$ and $(\bar{Y}) = 1/\mu$, where $\bar{X}$ and $\bar{Y}$ are the sample means of lifetimes and repair times of the system respectively. It can be shown that $\bar{X}$ and $\bar{Y}$ are the MLEs of $1/\lambda$ and $1/\mu$ respectively. Let $\theta_1 = 1/\lambda$ and $\theta_2 = 1/\mu$. Clearly, the expected number of operable units in the system given in section 3.1.3 reduces to $E(Q) = \frac{n}{1+\rho} = \frac{n\mu}{\lambda+\mu} = \frac{n\theta_1}{\theta_1+\theta_2} = Q_s$ (say). Hence, the MLE of Q_s is readily obtained by replacing θ_1 and θ_2 by the corresponding maximum likelihood estimators $\bar{X}$ and $\bar{Y}$ (by the invariance property of MLE).

$$\text{That is, } \hat{Q}_s = \frac{n\bar{X}}{\bar{X}+\bar{Y}} \, . \tag{3.9}$$

It may be noted that $\hat{Q}_s$ given in (3.9) is a real valued function in $\bar{X}$ and $\bar{Y}$, which is also differentiable. By applying the multivariate central limit theorem (see Radhakrishna Rao (1974)), we have $\sqrt{N}[(\bar{X},\bar{Y}) - (\theta_1,\theta_2)] \xrightarrow{d} N_2(0,\Sigma)$ as N → ∞, where the dispersion matrix $\Sigma = ((\sigma_{ij}))$ is given by $\Sigma = diag(\theta_1{}^2, \theta_2{}^2)$. Again from Radhakrishna Rao (1974), we have

$\sqrt{N}(\hat{Q}_s - Q_s) \xrightarrow{d} N[0,\sigma^2(\theta)]$ as N → ∞, where $\theta = (\theta_1,\theta_2)$ and

$$\sigma^2(\theta) = \sum_{i=1}^{2}\left(\frac{\partial Q_s}{\partial \theta_i}\right)^2 \sigma_{ii} = \left(\frac{\partial Q_s}{\partial \theta_1}\right)^2 \theta_1{}^2 + \left(\frac{\partial Q_s}{\partial \theta_2}\right)^2 \theta_2{}^2 = \frac{2n^2\theta_1{}^2\theta_2{}^2}{(\theta_1+\theta_2)^4} \tag{3.10}$$

Thus, $\hat{Q}_s$ is a CAN estimator of Q_s . There are several methods for generating CAN estimators and the Method of Moments and the Method of Maximum likelihood are commonly used to generate such estimators, See Sinha (1986).

Let $\sigma^2(\hat{\theta})$ be the estimator of $\sigma^2(\theta)$ obtained by replacing θ by a consistent estimator $\hat{\theta}$ namely $\hat{\theta} = (\bar{X},\bar{Y})$. Let $\hat{\sigma}^2 = \sigma^2(\hat{\theta})$. Since $\sigma^2(\theta)$ is a continuous function of θ, $\hat{\sigma}^2$ is a consistent estimator of $\sigma^2(\theta)$. That is, $\hat{\sigma}^2 \xrightarrow{P} \sigma^2(\theta)$ as N → ∞. By Slutsky theorem $\left(X_n \xrightarrow{d} X, Y_n \xrightarrow{P} b \Rightarrow \frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{b}, b \neq 0\right)$, we have, $\sqrt{N}\frac{(\hat{Q}_s - Q_s)}{\hat{\sigma}} \xrightarrow{d} N(0,1)$. That is, $Pr\left[-k_{\frac{\alpha}{2}} < \sqrt{N}\frac{(\hat{Q}_s - Q_s)}{\hat{\sigma}} < k_{\frac{\alpha}{2}}\right] = (1-\alpha)$, where $k_{\frac{\alpha}{2}}$ is obtained from normal tables.

Hence, a 100(1− α) % asymptotic confidence interval for Q_s is given by

$$\hat{Q}_s \pm k_{\frac{\alpha}{2}}.\frac{\hat{\sigma}}{\sqrt{N}} \quad , \qquad (3.11)$$

where $\hat{\sigma}$ is a consistent estimator of $\sigma(\theta)$ and is obtained from (3.10).

4. Bayesian analysis of an n unit system operating in a random environment

Measures of system performance of various systems in Reliability theory such as system reliability, MTBF, point availability, steady state availability and so on have been studied using their respective failure time and repair time density functions. The ratio of failure rate to service rate is an important parameter of these probability mass functions. Reliability theory is concerned with statistical description of the behaviour of various measures of system performance expressed in terms of this parameter. However, in all such studies, parametric estimation, interval estimation, testing of statistical hypothesis and Bayesian inferential aspects of these measures of system performance have not been studied. The failures and repairs in any system are greatly influenced by a number of factors such as system configuration, the environmental conditions under which the system operates and so on, which cannot be controlled or assessed well in advance and hence there is an urgent need to make use of the sample information in order to draw valid inferences about measures of system performance.

In this Section, Bayes estimator of ρ under squared error loss, the minimum posterior risk associated with Bayes estimator and the minimum Bayes risk of $\hat{\rho}^B$ are obtained using the same data as in Section 3. That is the number of units operable at several sampled time points. Beta distribution of second kind is taken as natural conjugate prior density for ρ.

4.1 Bayes estimator of ρ

Assume that ρ has a prior distribution Beta of second kind with the hyper parameters α and β.

That is $$\tau(\rho\,|\,\alpha,\beta) = \frac{1}{\beta_2(\alpha,\beta)}.\frac{\rho^{\alpha-1}}{(1+\rho)^{\alpha+\beta}},$$

$$0 < \rho < \infty; \quad \alpha, \beta > 0 \qquad (4.1)$$

The marginal p.d.f. of $Y = \sum_{i=1}^{N} X_i$, which is called the predictive p.d.f. is given by

$$f^*(y) = \int_0^\infty f(y;\rho)\,\tau(\rho|\alpha,\beta)d\rho$$

$$= \frac{1}{\beta_2(\alpha,\beta)}\int_0^\infty \binom{nN}{y}\left(\frac{1}{1+\rho}\right)^{y}\left(\frac{\rho}{1+\rho}\right)^{nN-y}.\frac{\rho^{\alpha-1}}{(1+\rho)^{\alpha+\beta}}d\rho$$

$$= \binom{nN}{y}\frac{\beta_2(nN+\alpha-y,\beta+y)}{\beta_2(\alpha,\beta)} \qquad (4.2)$$

Hence, the posterior distribution of ρ is given by

$$q(\rho|x_1,x_2,\dots,x_N) = \frac{f(y;\rho)\tau(\rho|\alpha,\beta)}{\int_0^\infty f(y;\rho)\;\tau(\rho|\alpha,\beta)d\rho}$$

$$= \frac{1}{\beta_2(nN+\alpha-y,\beta+y)}\frac{\rho^{(nN-y+\alpha)-1}}{(1+\rho)^{nN+\alpha+\beta}},$$

$$0 < \rho < \infty \qquad (4.3)$$

Remark 4.1

i. The posterior distribution of ρ is also that of Beta distribution of second kind with the parameters $(nN+\alpha-y,\beta+y)$

ii. The posterior p.d.f. of ρ reflects both the prior information (α, β) and the sample information $Y = \sum_{i=1}^{N} X_i$,

Now, the Bayes estimator of ρ under squared error loss is given by

$$E(\rho|x_1,x_2,\dots,x_N) = \int_0^\infty \rho q(\rho|x_1,x_2,\dots,x_N)d\rho$$

$$= \frac{1}{\beta_2(nN+\alpha-y,\beta+y)}\int_0^\infty \frac{\rho^{(nN+\alpha-y)}}{(1+\rho)^{nN+\alpha+\beta}}d\rho$$

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$$= \frac{(nN+\alpha-y)}{(\beta+y-1)} \quad . \tag{4.4}$$

Remark 4.2

It may be noted that

$$E(\rho|x_1, x_2, \dots, x_N) = \frac{(nN+\alpha-y)}{(\beta+y-1)}$$

$$= \frac{y}{(\beta+y-1)}\frac{(nN-y)}{y} + \frac{(\beta-1)}{(\beta+y-1)}\frac{\alpha}{(\beta-1)}$$

which is the weighted average of the maximum likelihood estimator $\frac{(nN-y)}{y}$ and the mean $\frac{\alpha}{(\beta-1)}$ of the prior pdf of the parameter ρ, where the respective weights are $\frac{y}{(\beta+y-1)}$ and $\frac{(\beta-1)}{(\beta+y-1)}$.

4.2 Minimum posterior risk associated with Bayes estimator

The minimum posterior risk associated with the Bayes estimator is given by

$$V_\rho\left(\hat{\rho}^B(x_1, x_2, \dots, x_N)\right) = E[\hat{\rho}-\rho]^2$$

$$= \int_0^\infty (\hat{\rho}-\rho)^2 q(\rho|x_1, x_2, \dots, x_N)d\rho$$

$$= \frac{(nN-y)(\beta+y-2)[(nN-y)(\beta-y-1)-2\alpha y] + y^2(nN+\alpha-y)(nN+\alpha-y+1)}{y^2(\beta+y-1)(\beta+y-2)} \tag{4.5}$$

4.3 Minimum Bayes Risk

The minimum Bayes risk $r_{\tau,\hat{\rho}^B}$ of $\hat{\rho}^B$ is given by

$$r_{\tau,\hat{\rho}^B} = E[V_\rho(\hat{\rho}^B|x_1, x_2, \dots, x_N)]$$

$$= \sum_{y=0}^{nN} V_\rho(\hat{\rho}^B|x_1, x_2, \dots, x_N)h(x_1, x_2, \dots, x_N) \ , \tag{4.6}$$

where $h(x_1, x_2, \dots, x_N)$ is the marginal distribution of $(x_1, x_2, \dots, x_N)$ and is given by

$$h(x_1, x_2, \dots, x_N) = \int_0^\infty L(\rho|x_1, x_2, \dots, x_N)\tau(\rho|\alpha,\beta)d\rho$$

$$= \frac{1}{\beta_2(\alpha,\beta)}\int_0^\infty \left\{\prod_{i=1}^{N}\binom{n}{x_i}\right\}\left(\frac{1}{1+\rho}\right)^y\left(\frac{\rho}{1+\rho}\right)^{nN-y} \cdot \frac{\rho^{\alpha-1}}{(1+\rho)^{\alpha+\beta}}d\rho$$

$$= \frac{\beta_2(nN+\alpha-y,\beta+y)}{\beta_2(\alpha,\beta)}\prod_{i=1}^{N}\binom{n}{x_i} \quad . \tag{4.7}$$

Substituting (4.5) and (4.7) in (4.6), we get

$$r_{\tau,\hat{\rho}^B} = \frac{\prod_{i=1}^{N}\binom{n}{x_i}}{\beta_2(\alpha,\beta)}\sum_{y=0}^{nN}\left\{\frac{(nN-y)(\beta+y-2)[(nN-y)(\beta-y-1)-2\alpha y] + y^2(nN+\alpha-y)(nN+\alpha-y+1)}{y^2(\beta+y-1)(\beta+y-2)}\right\}X$$

$$\beta_2(nN+\alpha-y, \beta+y)$$

In the following Section 5, we provide a numerical illustration to study the performance of $\hat{Q}_S$.

5. Numerical illustration

The CAN estimator of Q_s is obtained and its performance is studied. Here 1000 random samples are generated independently 50 times from the exponential distributions assuming λ=3 and μ=4. The CAN estimator of Q_s namely $\hat{Q}_s$ is obtained using these estimates. Table 1 shows the calculated values of $\hat{Q}_s$ and 95 % lower and upper confidence limits.

The Mean Square Error is obtained by using the formula $\frac{1}{50}\sum(\hat{Q}_s - Q_s)^2$ to find the performance of CAN estimator of Q_s. The Mean Square Error is found to be 0.1596421. Hence we conclude that the proposed estimator performs reasonably well.

Table 1 Values of n (no. of units in the system), $\hat{Q}_s$, $\hat{\sigma}$ and 95 % confidence limits of Q_s

S.No.	n	$\hat{Q}_s$	$\hat{\sigma}$	Lower limit	Upper limit
1	11	6.937	7.102	6.713	7.162
2	12	7.626	7.705	7.382	7.870
3	13	8.232	8.369	7.967	8.496
4	14	8.600	9.195	8.309	8.891
5	15	9.219	9.848	8.908	9.530
6	16	10.025	10.377	9.697	10.354
7	17	10.718	10.978	10.371	11.065
8	18	11.387	11.596	11.020	11.754

9	19	11.706	12.456	11.312	12.100
10	20	13.075	12.549	12.678	13.472
11	21	13.173	13.609	12.742	13.603
12	22	13.437	14.497	12.978	13.895
13	23	14.280	15.007	13.805	14.754
14	24	14.882	15.672	14.386	15.377
15	25	15.620	16.245	15.106	16.134
16	26	16.483	16.724	15.954	17.012
17	27	16.379	17.859	15.814	16.943
18	28	17.419	18.246	16.842	17.996
19	29	18.040	18.898	17.443	18.638
20	30	18.975	19.329	18.364	19.586
21	31	19.583	19.991	18.951	20.216
22	32	19.534	21.093	18.866	20.201
23	33	20.939	21.213	20.268	21.610
24	34	21.709	21.753	21.021	22.397
25	35	22.334	22.403	21.625	23.042
26	36	22.515	23.377	21.775	23.254
27	37	23.461	23.796	22.709	24.214
28	38	22.901	25.223	22.103	23.699
29	39	24.663	25.131	23.868	25.457
30	40	24.418	26.366	23.584	25.251
31	41	26.321	26.121	25.495	27.147
32	42	25.763	27.607	24.890	26.636
33	43	26.706	28.050	25.819	27.593
34	44	28.006	28.218	27.114	28.899
35	45	28.987	28.591	28.083	29.891
36	46	29.064	29.661	28.126	30.002
37	47	29.051	30.752	28.078	30.023
38	48	29.889	31.259	28.901	30.878
39	49	30.601	31.850	29.594	31.608
40	50	30.498	32.972	29.455	31.541
41	51	31.395	33.452	30.337	32.453
42	52	31.994	34.119	30.915	33.073
43	53	33.223	34.363	32.137	34.310
44	54	33.536	35.227	32.422	34.650
45	55	33.091	36.538	31.935	34.246
46	56	35.223	36.224	34.077	36.368
47	57	35.351	37.216	34.174	36.528
48	58	35.912	37.909	34.713	37.111
49	59	37.044	38.211	35.836	38.252
50	60	37.460	39.007	36.227	38.694

From Table 1, it is evident that as the number of units in the system increases, the expected number of operable units in the system too increases and consequently the corresponding confidence intervals show an increasing trend.

6. Conclusion

An n unit system operating in a random environment is considered. Assuming that the environment is described by a Markov Process, a consistent estimator, MLE and Bayes estimator of ρ of the system based on the number of units operable at several sampled time points are obtained. Also, CAN estimator and asymptotic confidence limits for the expected number of operable units in the system are obtained. Further, simulation study is carried out to obtain the mean square error to assess the performance of CAN estimator of Q_s and concluded that the proposed estimator of the expected number of operable units in the system $\hat{Q}_S$ performed reasonably well.

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